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"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.

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"Meditationis est perscrutari occulta; contemplationis est admirari perspicua . . . . . Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

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—"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas;
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

*J. B. Pinelli ad Mazonium.*

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[FOURTH SERIES.]

*JANUARY* 1872.

## I. *Electrodynamic Measurements. By* Professor WILHELM WEBER.—Sixth Memoir, *relating specially to the Principle of the Conservation of Energy**.

THE law of electrical action announced in the First Memoir on Electrodynamic Measurements (*Elektrodynamische Maassbestimmungen*, Leipzig, 1846) has been tested on various sides and been modified in many ways. It has also been made the subject of observations and speculations of a more general kind; these, however, cannot by any means be regarded as having as yet led to definite conclusions. The First Part of the following Memoir is limited to a discussion of the relation which this law bears to the *Principle of the Conservation of Energy*, the great importance and high significance of which have been brought specially into prominence in connexion with the Mechanical Theory of Heat. In consequence of its having been asserted that the law referred to is in contradiction with this principle, an endeavour is here made to show that no such contradiction exists. On the contrary, the law enables us to make an addition to the Principle of the Conservation of Energy, and to alter it so that its application to each pair of particles is no longer limited solely to the time during which the pair does not undergo either increase or diminution of *vis viva* through the action of other bodies, but always holds good independently of the manifold relations to other bodies into which the two particles can enter.

Besides this, in the Second Part the law is applied to the de-

* Translated by Professor G. C. Foster, F.R.S., from the *Abhandlungen der mathem.-phys. Classe der Königl. Sächsischen Gesellschaft der Wissenschaften*, vol. x. (January 1871).

velopment of *the equations of motion of two electrical particles subjected only to their mutual action.* Albeit this development does not lead directly to any comparisons or exact control by reference to existing experience (on which account it has hitherto received little attention), it nevertheless leads to various results which appear to be of importance as furnishing clues for the investigation of the molecular conditions and motions of bodies which have acquired such special significance in relation to Chemistry and the theory of Heat, and to offer to further investigation interesting relations in these still obscure regions.

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## On the Relation between the Laws of Electricity and the Principle of the Conservation of Energy.

### 1. *Electrical Particles and Electrical Masses.*

Particles of positive and of negative electricity are denoted by the same letters, for instance by $e$ or $e'$ &c., but a positive or a negative value is assigned to $e$ or $e'$ ... according to whether it represents a particle of the positive or of the negative fluid.

If the measurable force of repulsion exerted by the first particle $e$ upon another exactly equal particle $e$ at the constant measurable distance $r$ be denoted by $f$, and also the measurable force of repulsion exerted by the second particle $e'$ upon another exactly equal particle $e'$, at the same distance $r$, be denoted by $f'$, then $\pm r\sqrt{f}$ is taken as the measure of $e$, and $\pm r\sqrt{f'}$ as the measure of $e'$, where the upper or the lower sign is to be taken according to whether the particle is a particle of positive or of negative fluid. The unit of force which is here adopted for the measurement of $f$ and $f'$ is the unit recognized in Mechanics, namely the force which, when it acts upon the unit of mass recognized in Mechanics (1 milligramme), imparts to this mass unit of velocity in unit of time. The repulsive force of the two particles $e, e'$, so long as their distance $r$ remains unchanged, is, in acccordance with the electrostatical law,

$$=\frac{ee'}{rr}.$$

A negative value of this expression denotes attractive force.

In this mode of denoting particles of the electric fluids, however, $e, e'$ have not the signification of *masses* in the mechanical sense, as appears from the simple consideration that $e, e'$ may have at one time positive and at another time negative values; but nevertheless the values of $e, e'$ are closely related to the masses of the particles. For if we denote the *masses* of the particles $e, e'$ (in the mechanical sense, according to which the unit of mass [1 milligramme] is determined by the mass of *one* ponderable body, and different masses are compared with each other in proportion to the reciprocals of the accelerations produced in them by the same force) by $\epsilon$, $\epsilon'$, of which the values are always positive, we get for *positive* values of $e$, $e'$,

$$\frac{e}{\epsilon}=\frac{e'}{\epsilon'}=a;$$

and for negative values of $e$, $e'$,

$$\frac{e}{\epsilon}=\frac{e'}{\epsilon'}=b,$$

where $a$ has a definite *positive* and $b$ a definite *negative value.* Whether or not we have here $aa=bb$, or what ratio $aa$ bears to $bb$, has not as yet been made out, any more than the numerical value of $a$ or $b$. In many cases the electrical mass $\epsilon$ is connected with a ponderable mass $m$, so that it is impossible for it to be moved independently of it; in such cases, only the combined mass $m+\epsilon$ comes into account, and in general $\epsilon$ may be regarded as vanishingly small in comparison with $m$. Consequently it is only seldom that the masses $\epsilon$, $\epsilon'$ have to be considered.

The distinction here indicated between the particles $e$, $e'$ and their masses $\epsilon$, $\epsilon'$ is not always made; on the contrary, the symbols of the particles $e$, $e'$ are also used to denote the corresponding masses. It is, however, to be observed that, when this is done, no regard can be had to the signs of $e$, $e'$. The omission of the unknown factors $a$ and $b$ is always allowable when we are dealing only with the *relative values* of masses of positive or of negative electricity.

## 2. *The Law of Electrical Force.*

The Law of Electrical Force is thus stated in 'Electrodynamic Measurements' (Leipzig, 1846, p. 119:—

> If $e$ and $e'$ denote two electrical particles, the repulsive force exerted by the two particles on each other at the distance $r$ is represented by
>
> $$\frac{ee'}{rr}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2r}{cc}\frac{ddr}{dt^2}\right),$$

where $c$ is the constant denoted at the place quoted by $\frac{4}{a}$.

But this expression for the force which the particles $e$ and $e'$ mutually exert upon each other, it is easy to see, is dependent on a magnitude which contains as a factor the very force that is to be determined. This is readily seen when the relative acceleration of the two particles, namely $\frac{ddr}{dt^2}$, is broken up into two parts, thus,

$$\frac{ddr}{dt^2}=\frac{ddr'}{dt^2}+\frac{ddr''}{dt^2};$$

where the first part, $\frac{ddr'}{dt^2}$, is the relative acceleration due to the mutual action of the two particles, and the second part, $\frac{ddr''}{dt^2}$, is the acceleration due to other causes (namely to the acquired velocity of the particles perpendicular to $r$, and to the mutual action between them and other bodies). The first part, however, or that due to the mutual action of the two particles, is proportional to the force arising from this mutual action, and is represented by the quotient of this force by the mass upon which it acts.

Hence there easily follows, as was shown in the memoir already quoted (page 168), another expression for the force which the particles $e$ and $e'$ mutually exert upon each other, containing only terms which are independent of the force to be determined, namely the expression

$$\frac{ee'}{rr-\frac{2r}{cc}(e+e')}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2rf}{cc}\right)$$

(in which $f$ is put for $\frac{ddr''}{dt^2}$), or, if the electrical particles $e$ and $e'$ are distinguished from their masses $\epsilon$ and $\epsilon'$ in accordance with the previous section (a distinction which was not made in the memoir quoted above), the expression

$$\frac{ee'}{rr-\frac{2r}{cc}\cdot\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}ee'}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2rf}{cc}\right).$$

From this it results that the law of electrical force is by no means so simple as we expect a fundamental law to be; on the contrary, it appears in two respects to be particularly complex.

In the first place, it is a consequence of this expression for the force, that, as was pointed out in the memoir referred to, the force which two electrical particles exert upon each other does not depend exclusively upon these particles themselves, but also upon the portion of their relative acceleration denoted by $f$, which is in part due to the action of other bodies. It was also pointed out that, inasmuch as the forces exerted by two bodies upon each other have been called by Berzelius *catalytic forces* when they depend upon the presence of a third body, electrical forces considered generally are, in this sense, catalytic forces.

In the second place, another noteworthy result follows from this expression for the force—namely, that when the particles $e$ and $e'$ are of the same kind, *they do not by any means always* *repel each other*; thus when $\frac{dr^2}{dt^2} < cc + 2rf$, they repel only so long as $r > \frac{2}{cc} \frac{\epsilon + \epsilon'}{\epsilon\epsilon'} ee'$, and, on the contrary, they attract when

$$r < \frac{2}{cc} \frac{\epsilon + \epsilon'}{\epsilon\epsilon'} ee'.$$

An exception to this rule occurs only in the case in which $\left(r - 2\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \frac{ee'}{cc}\right)$, which is always a factor of the denominator, becomes likewise a factor of the numerator. This case occurs when the two electrical particles are at *permanent relative rest*, so that $\frac{dr}{dt} = 0$ and $\frac{ddr}{dt^2} = 0$.

The general expression for the force given above becomes in fact

$$\frac{ee'}{r\left(r - 2\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc} f\right)$$

when $\frac{dr}{dt} = 0$; and by dividing this by the mass $\frac{\epsilon\epsilon'}{\epsilon + \epsilon'}$, we find the part of the acceleration which is due to the forces exerted upon each other by the two electrical particles, namely

$$\frac{(\epsilon + \epsilon')ee'}{\epsilon\epsilon' r\left(r - 2\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc} f\right).$$

By adding to this the other part of the acceleration, namely $f$, which is due to the acquired motion of the particles at right angles to $r$ and to the action of other bodies, we obtain the *total* acceleration, namely

$$\frac{ddr}{dt^2} = f + \frac{(\epsilon + \epsilon')ee'}{\epsilon\epsilon' r\left(r - 2\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc} f\right),$$

which, when the particles are at permanent relative rest, $= 0$. Hence for permanent relative rest we have

$$f = -\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{rr}.$$

If this value of $f$ be substituted in the expression for the force

$$\frac{ee'}{r\left(r - 2\frac{\epsilon + \epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc} f\right),$$

the latter becomes

$$\frac{ee'}{r\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right)}\cdot\frac{1}{r}\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right).$$

Hence it appears that, in the case of permanent relative rest, the factor $\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right)$ is common to numerator and denominator. The value of the quotient, which is thus independent of this factor, namely $\frac{ee'}{rr}$, consequently gives the expression for the force, in the case of permanent relative rest, in complete agreement with the fundamental laws of electrostatics, according to which this force has a *positive* value for particles of the *same kind* at *all distances.*

### 3. *The Law of Electrical Potential.*

In the previous section the law of electrical force is shown to be, in two respects, of a very complicated character, namely:—in the first place, in that the repulsive force between two electrical particles is dependent on things that do not appertain either to the nature of the particles which exert the force upon each other, or to their relative positions in space, or their existing relative motion, but *depends upon other bodies*; and secondly, in that *repulsion* may be exerted upon each other at certain distances by the same particles, and *attraction* at other distances.

Compared with this complicated law of *electrical force,* the law of *electrical potential* is very simple.

The value of the potential V of two electrical particles $e$ and $e'$, in fact, as I pointed out as long ago as the year 1848 in Poggendorff's *Annalen* (vol. lxxiii. p. 229), is determined by the following law,

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right).$$

Observing that both $r$ and $\frac{dr}{dt}$ have different values at different times for both the particles $e$ and $e'$, and that consequently both are functions of the time, it follows that $\frac{dr}{dt}$ may also be regarded as a function of $r$, which may be denoted by $fr$. We thus obtain

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot(fr)^2-1\right),$$

and from this, by differentiation, the expression for the force

$$\frac{dV}{dr} = -\frac{ee'}{rr}\left(\frac{1}{cc}\cdot(fr)^2 - 1\right) + 2\frac{ee'}{rcc}\cdot fr\cdot\frac{dfr}{dr},$$

or, if we again put $\frac{dr}{dt}$ for $fr$,

$$\frac{dV}{dr} = \frac{ee'}{rr}\left(1 - \frac{1}{cc}\cdot\frac{dr^2}{dt^2} + \frac{2r}{cc}\cdot\frac{dr}{dt}\cdot\frac{d\frac{dr}{dt}}{dr}\right),$$

for which we may write

$$\frac{dV}{dr} = \frac{ee'}{rr}\left(1 - \frac{1}{cc}\,\frac{dr^2}{dt^2} + \frac{2r}{cc}\cdot\frac{ddr}{dt^2}\right).$$

From this it appears that

$$\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$$

is a function whose differential coefficient with respect to $r$ represents the repulsive force between the two particles $e$ and $e'$, where $r$ and $\frac{dr}{dt}$ denote respectively their distance and relative velocity regarded as functions of the time. But since $\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$ becomes equal to nothing when $e$ and $e'$ are separated infinitely far from each other, $\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$ is the *potential* of the electrical particles $e$ and $e'$—that is to say, the *work* which is expended in causing the particles to approach each other from an infinite distance while under the action of their mutual repulsion, and to arrive at the distance $r$ with the relative velocity $\frac{dr}{dt}$*.

It likewise results from the foregoing that the *work*, which is expended when a given relative arrangement and state of motion of a system of particles $e$, $e'$ are changed to another arrangement and another state of motion, depends only on the initial and

* This law of electrical *potential* has also been taken as his starting-point by Beer in his 'Introduction to Electrodynamics' (see *Einleitung in die Elektrostatik, die Lehre vom Magnetismus und die Elektrodynamik*, von August Beer. Nach dem Tode des Verfassers herausgegeben von Julius Plücker: Braunschweig, 1865. S. 250). The placing of the law of *potential* in the foreground as the fundamental law, and deriving the law of force from it, ought not to give rise to any misgiving. We have in many respects a better justification for speaking of the *physical existence of the work expressed by the potential* than for speaking of the *physical existence of a force*, as to which all we can say is that it *tends to change the physical relations of bodies.*

final arrangements and movements of the particles, and is independent of the way by which the transition has been effected, and also independent of states of motion which may have existed during the transition.

## 4. *Fundamental Electrical Laws.*

The law of *electrical potential* certainly appears to stand, in view of its simplicity, in a much closer relation to the true fundamental laws of electricity than the far more complex law of *electrical force*; but the expression of the former law may still be resolved into two simpler laws, which may be stated in the following manner:—

*First Law.*—If two particles $e$ and $e'$ are at relative rest or possess the same relative motion at two different distances $r$ and $\rho$, the quantities of work V and U which are expended in separating the particles, while mutually acting on each other, from these distances to an infinite distance, are to each other inversely as these two distances, that is,

$$V:U=\rho:r. \qquad (1)$$

*Second Law.*—The work U, which is expended in separating the particles $e$ and $e'$ while subject to the force exerted by them on each other from *a given distance* $\rho \left(=\frac{ee'}{a}\right)$ *proportional to the quantity* $ee'$ to an infinite distance, makes *together with the vis viva* x, which belonged to the particles in consequence of their relative motion at the distance $\rho$, *a constant sum*, namely $a$; that is,

$$U+x=a. \qquad (2)$$

For from equation (1) it follows that

$$U=\frac{r}{\rho}V;$$

and hence, by equation (2),

$$\frac{r}{\rho}V+x=a,$$

or, since $\rho=\frac{ee'}{a}$,

$$V=\frac{ee'}{r}\left(1-\frac{x}{a}\right).$$

But the relative *vis viva* $x$ is proportional to the square of the relative velocity $\frac{dr}{dt}$, so that we may substitute for $a$ a new constant *cc*, such that

$$\frac{x}{a}=\frac{1}{cc}\cdot\frac{dr^2}{dt^2}*.$$

* If $\epsilon$ and $\epsilon'$ denote the masses of the particles $e$ and $e'$, $\alpha$ and $\beta$ the velocities of $\epsilon$ in the direction of $r$ and at right angles thereto, and $\alpha'$ and $\beta'$ the same velocities for $\epsilon'$, so that $\alpha-\alpha'=\frac{dr}{dt}$ is the relative velocity of the two particles, then

$$\tfrac{1}{2}\epsilon(\alpha\alpha+\beta\beta)+\tfrac{1}{2}\epsilon'(\alpha'\alpha'+\beta'\beta')$$

is the total *vis viva* of the two particles. If we now put for $\alpha$

$$\frac{\epsilon\alpha+\epsilon'\alpha'}{\epsilon+\epsilon'}+\frac{\epsilon'(\alpha-\alpha')}{\epsilon+\epsilon'},$$

and for $\alpha'$

$$\frac{\epsilon\alpha+\epsilon'\alpha'}{\epsilon+\epsilon'}-\frac{\epsilon'(\alpha-\alpha')}{\epsilon+\epsilon'},$$

we get the total *vis viva* of the two particles represented as the sum of two parts in the following manner—namely,

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}+\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+\epsilon\beta\beta+\epsilon'\beta'\beta'\right],$$

the *first* part of which, or $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, is the *relative* vis viva of the particles which was denoted above by $x$. But $a$ is also a relative *vis viva* of the same particles, namely that which corresponds to a definite relative velocity $c$, so that $a=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot cc$. Hence we get $\frac{x}{a}=\frac{1}{cc}\cdot\frac{dr^2}{dt^2}$, as was given above.

It may be further observed that the *second* part of the above sum, namely $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+\epsilon\beta\beta+\epsilon'\beta'\beta'\right]$, may be again represented, after another subdivision, as the sum of two parts, thus

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{ds^2}{dt^2}+\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right],$$

where $\frac{ds}{dt}$ represents the velocity with which the two particles move relatively to each other in space perpendicularly to $r$, while $\gamma$ represents the velocity, perpendicular to $r$, of the centre of gravity of the two particles. We thus get the total *vis viva* of the two particles divided into three parts—namely,

i. $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$,

ii. $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{ds^2}{dt^2}$,

iii. $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right]$;

the *first* of which, namely $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, represents the *relative* vis viva of

We thus obtain

$$V=\frac{ee'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right).$$

Here V denotes the work expended in separating the two particles from the distance $r$ to an infinite distance. If V is to denote the work done in bringing the particles from an infinite distance to the distance $r$, as it is usually understood to do, so that positive values of $\frac{dV}{dr}$ may indicate *repulsion*, we obtain

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right);$$

that is to say, the *law of electrical potential.*

### 5. *Principle of the Conservation of Energy for Two Particles which form a detached system.*

The two fundamental laws laid down in the foregoing section, which may be called

The Law of the dependence of the Potential on the distance for *a constant relative motion*, and

The Law of the dependence of the Potential on the relative motion for *a constant distance*,

require to be further discussed in relation to their bearing upon the principle of the Conservation of Energy.

In accordance with the principle of the conservation of energy, three forms of energy are to be distinguished from each other—namely, *energy of motion* (kinetic energy), *potential energy*, and *energy of heat* (thermal energy).

The *energy of motion* is that part of the energy which depends upon the existing movements; and a special determination is given of the way in which it depends upon movement—namely, partly upon the magnitude of the moving mass, and partly upon the velocity with which this mass moves.

The same determination also appies to *thermal energy*, if this is regarded, in accordance with the mechanical theory of heat,

---

the two particles; while the *first* two parts taken together, namely

$$\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\left(\frac{dr^2}{dt^2}+\frac{ds^2}{dt^2}\right),$$

represent the total *internal* vis viva, or the total *internal kinetic energy of the system*; and the *third* part, namely $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right]$, represents the *external* vis viva, or the *external kinetic energy of the system* (that is, the *vis viva* of the centre of gravity of the two particles).

as an *internal motion of the particles of bodies.* But if we are dealing with a system of two *elementary particles* (that is to say, particles such that there can be no motion *within* them), it is obvious that in the case of such a system thermal energy has no existence, and *energy of motion* and *potential energy* alone remain.

Lastly, the *potential energy* is that part of the energy which depends on the existing potential; and a special determination is needed of the way in which potential energy *depends upon the potential,* exactly as, in the case of the energy of motion, it is needful to determine the special way in which it depends on movement.

Now this special determination has been made by *equating potential energy* (without regard to the sign) *and potential*.*

The justification for this proceeding has been found in the fact that the potential is a magnitude which is homogeneous with kinetic energy, which, when taken with the negative sign and added to the kinetic energy, gives always the same sum, so long as the two particles constitute a detached system which does not undergo either gain or loss of energy from without.

For instance, if we have a system of two ponderable particles $m, m'$, its *potential* is

$$V=\frac{mm'}{r};$$

and the internal *vis viva,* or the *internal kinetic energy of the system,* is

$$W=\frac{1}{2}\,\frac{mm'}{m+m'}\,(uu+\alpha\alpha),$$

where $u=\frac{dr}{dt}$ is the relative velocity of the two particles, and $\alpha$ the difference of the velocities in space perpendicularly to $r$. But, for such a *detached system,* if we put $r=r_0$ and $\alpha=\alpha_0$

* The sign of the *potential,* V, is so determined that positive values of $\frac{dV}{dr}$ indicate repelling forces; the sign of the *potential energy* is fixed by the sign of the work which is done, in consequence of the mutual action of the particles, when the two particles are separated from the distance $r$ to an infinite distance. Consequently, for two ponderable particles $m, m'$, the potential is $V=\frac{mm'}{r}$, and the potential energy $=-\frac{mm'}{r}$. For two electrical particles $e, e'$ the potential is $=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right)$, and the potential energy $=\frac{ee'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right)$.

when $u=0$, the following value is easily got, namely

$$uu=\frac{r_0-r}{r_0}\left[\frac{2(m+m')}{r}-\frac{r_0+r}{r_0}\alpha\alpha\right]^*,$$

and consequently the sum

$$W-V=-\frac{mm'}{r_0}+\frac{1}{2}\frac{mm'}{m+m'}\cdot\alpha_0\alpha_0.$$

This sum always retains the same value as long as the values of $r_0$ and $\alpha_0$ remain unchanged—that is, so long as the system of the two particles undergoes neither loss nor gain of energy from without. The *external kinetic energy* of such a detached system amounts *separately to a constant sum.*

Now the same thing holds good also for two *electrical* particles $e, e'$; for their potential, taken with the negative sign and added to their kinetic energy, gives in like manner always the same sum so long as the particles constitute a *detached* system.

* The force with which the two particles mutually act on each other, namely $\frac{dV}{dr}$, divided by $m$, gives the acceleration of the particle $m$—that is, $\frac{1}{m}\cdot\frac{dV}{dr}$; divided by $m'$ it gives the acceleration of the particle $m'$, namely $\frac{1}{m'}\cdot\frac{dV}{dr}$. Consequently that part of the relative acceleration of the two particles which arises from their mutual action is $\left(\frac{1}{m}+\frac{1}{m'}\right)\frac{dV}{dr}$, while that part of the relative acceleration of the two particles which arises from their rotation about one another is represented by $\frac{\alpha\alpha}{r}$. If now this last portion be subtracted from the total acceleration $\frac{du}{dt}$, the following equation results:

$$\frac{du}{dt}-\frac{\alpha\alpha}{r}=\left(\frac{1}{m}+\frac{1}{m'}\right)\frac{dV}{dr}.$$

Putting $r=r_0$ and $\alpha=\alpha_0$ for the instant at which $u=0$, we obtain the expression

$$\alpha r=\alpha_0 r_0$$

as applicable for the case in which the only forces acting on the two particles are those due to their mutual action. Accordingly we get, by integrating the above differential equation after it has been multiplied by $2dr=2udt$,

$$uu+\alpha_0\alpha_0 r_0 r_0\left(\frac{1}{rr}-\frac{1}{r_0 r_0}\right)=2\left(\frac{1}{m}+\frac{1}{m'}\right)\left(\frac{mm'}{r}-\frac{mm'}{r_0}\right),$$

and hence

$$uu=\frac{r_0-r}{r}\left(\frac{2(m+m')}{r_0}-\frac{r_0+r}{r}\alpha_0\alpha_0\right)=\frac{r_0-r}{r_0}\left(\frac{2(m+m')}{r}-\frac{r_0+r}{r_0}\alpha\alpha\right).$$

We have, for the *potential* of such a system of two electrical particles,

$$V = \frac{ee'}{r}\left(\frac{uu}{cc} - 1\right),$$

and, for the *internal kinetic energy of the system,*

$$W = \frac{1}{2}\,\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\,(uu+\alpha\alpha) = \frac{ee}{\rho cc}\,(uu+\alpha\alpha),$$

if $u = \frac{dr}{dt}$ denotes the relative velocity of the two particles, and $\alpha$ the difference of their velocities in space at right angles to $r$. But, for such a *detached* system, when we put $r = r_0$ and $\alpha = \alpha_0$ for $u = 0$, it is easy to obtain

$$\alpha = \frac{r_0}{r}\,\alpha_0,$$

$$uu = \frac{r - r_0}{r - \rho}\left(\frac{\rho}{r_0}\,cc + \frac{r_0 + r}{r}\,\alpha_0\alpha_0\right)^*,$$

and consequently the sum

$$W - V = \frac{ee'}{r_0} + \frac{ee'}{\rho}\cdot\frac{\alpha_0\alpha_0}{cc} = \frac{ee'}{r_0} + \frac{1}{2}\,\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\alpha_0\alpha_0.$$

This sum likewise retains the same value so long as the values of $r_0$ and $\alpha_0$ remain unchanged—that is, so long as the system of two particles undergoes neither loss nor gain of energy from without†. *The same principle holds good in relation to the external kinetic energy of a detached system of two electrical particles and to that of two ponderable particles.*

* See Section 11.

† In Professor Tait's very instructive work, 'A Sketch of Thermodynamics' (Edinburgh, 1868), the following passage occurs at page 76, in reference to the investigations of Riemann and Lorenz which appeared in Poggendorff's *Annalen* for 1867 [Phil. Mag. S. 4. vol. xxxiv. pp. 368 and 287]:—"But the investigations of these authors are entirely based on Weber's inadmissible theory of the forces exerted on each other by *moving electric particles*, for which the conservation of energy is not true, while Maxwell's result is in perfect consistence with that great principle." This assertion of Professor Tait's seems to be in contradiction with the above. At page 56 of the same work Mr. Tait mentions that Helmholtz has based the doctrine of energy on Newton's principle and on the following postulate:—"Matter consists of ultimate particles which exert upon each other forces whose directions are those of the lines joining each pair of particles, and whose magnitudes depend solely on the distances between the particles." The contradiction between the fundamental law of electricity and *this postulate* is evident; but the contradiction between it and the *principle of the conservation of energy* is by no means evident,—a distinction which Professor Tait seems to have overlooked.

6. *Extension of the Principle of the Conservation of Energy to two electrical particles which do not form a detached system.*

If potential energy is taken, as is done in the previous section, as equal and opposite to potential, the principle of the conservation of energy holds good for two particles only so long as these two particles constitute a *detached* system—that is, so long as the system formed of the two particles undergoes neither gain nor loss of energy from without.

If the *total* energy of such a detached system of two particles were at first $=A$, but, the system ceasing to be detached, it received from without a quantity of kinetic energy $=a$, it seems to follow that, if the system were now again to become detached, the *total* energy would again become and remain constant so long as it remained detached, but that the total energy of the system in its final detached state would have the value $A+a$ (that is, a value exceeding that corresponding to its previous detached state by $a$). This, however, does not by any means conclusively prove the impossibility of extending the principle of the conservation of energy to two electrical particles which do not constitute a detached system.

For, strictly speaking, this has only been proved on the assumption that the *potential energy* of the system depends solely on the *distance* between the two particles; while if, on the other hand, the potential energy does not depend simply on the distance of the two particles, but also on their relative *motion*, it is evident that while the system receives from without an amount of *kinetic energy* $=a$, a change in its *potential energy* may be indirectly produced thereby. It is thus possible that the change of *potential energy*, so caused indirectly from without, might be $=-a$, so that the *total* energy (kinetic energy and potential energy together) of the two particles, even if they did not constitute a detached system, would retain always the same value.

This, however, certainly does not occur in reality for a system of two electrical particles, if the *potential energy* is taken as *equal and opposite to the potential*; but this assumption, which would thus make the extension of the principle impossible, has by no means been proved to be a necessary one. In general, all that is required is a *special determination of the way in which the potential energy depends upon the potential*; and here all that is self-evident is, that inasmuch as potential and potential energy are homogeneous magnitudes, a purely numerical relation must exist between them. But whether this numerical relation is always that of $+1$ to $-1$, or whether it is to be fixed otherwise, must still be regarded as in general doubtful; so that the possibility of the extension of the principle still remains.

We understand, in fact, by the *potential* of two particles, the amount of *work* which, in consequence of the mutual action of the two particles, is done when they are transferred in any way whatever from an infinite distance to the actually existing distance $r$ with the actually existing relative velocity $\frac{dr}{dt}$.

It is, however, evident that *work* is done, in consequence of the mutual action of the two particles, not only during their transference from a *greater* distance to the distance $r$, but also during their transference from a *smaller* distance to the distance $r$. And there is no obvious reason why the *energy ascribed to the system* should be made to depend on the work done in the *former* case, and not on that done in the *latter* case also.

For example, if the *first* quantity of work were denoted, according to Section 4, by V, and the *second* by $\frac{\rho-r}{\rho}V$, the potential energy ascribed to the system might be taken as the *difference of these two amounts of work*, namely $=\frac{\rho-r}{\rho}V-V=-\frac{r}{\rho}V$. This difference of the two amounts of work is evidently the quantity of work which is done, in consequence of the mutual action of the two particles, during their transference from the limiting value of *small* distances to the limiting value of *great* distances—that is to say, the value which $-V=\frac{ee'}{r}\left(1-\frac{uu}{cc}\right)$ assumes when $r$ is taken therein as equal to the limiting value of *small* distances, or when we put $r=\rho$, where $\rho$ denotes the limiting value of small distances. According to this, therefore, this *difference of the two quantities of work* $=\frac{ee'}{\rho}\left(1-\frac{uu}{cc}\right)=-\frac{r}{\rho}V$.

In order to determine in this way the potential energy of a system of two electrical particles when the *first* quantity of work above referred to is

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

it is only necessary further, for the determination of the *second* quantity of work, to determine the value of $\rho$—that is, of the *smaller distance* which is to be taken account of in that portion of the work.

Now this *smaller distance*, equally with the *greater distance*, must be determined *on its own account, independently of the actually existing conditions* of the two particles. This was done in the case of the *greater* distance by assigning to it an infinitely great value; in the case of the *smaller* distance the same thing is accomplished if we assign to it the value $2\frac{\epsilon+\epsilon'}{\epsilon\epsilon}\cdot\frac{ee'}{cc}$, a distance which is given by the particles $e, e'$, by their masses $\epsilon, \epsilon'$, and by the known electrical constant $c$.

If we now put the smaller distance equal to the value of $\rho$, we get, in virtue of the equations

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

$$\frac{\rho-r}{\rho}V=\frac{\rho-r}{\rho}\cdot\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

the required value of the *potential energy,* namely

$$-\frac{r}{\rho}V=-\frac{ee'}{\rho}\left(\frac{uu}{cc}-1\right)=\frac{1}{2}\frac{\epsilon\epsilon}{\epsilon+\epsilon'}(cc-uu).$$

In accordance with the distinction which is here drawn between the *potential* and the *potential energy* of two electrical particles and with the corresponding determination of their relation to each other, an analogous distinction may also be made between the *vis viva* and the *kinetic energy* of two particles. For there is no necessity that the *kinetic energy* of two particles should be taken as being equal to *the total* vis viva *of the two particles*; all that is generally essential is a *definite determination of the relation subsisting between the kinetic energy of two particles and the total* vis viva *belonging to them both.*

Now the total *vis viva* possessed by the two particles was represented in the note to section 4 as the sum of two parts, of which the *first* part, namely $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, was called the *relative* vis viva. The *second* part was that which the two particles possessed in virtue of their revolution about each other in space, and in virtue of the motion of their centre of gravity in space.

If now, in order to establish the conception of the *energy* of two particles, we take it as our starting-point that the *principle of the conservation of energy* of two particles must be based upon the essential characters of the two particles, and in fact upon what is *essential to them when regarded as constituting a detached system,* it is obvious that for this purpose the conception of the *energy* of two particles must be made to depend only on the relations presented by the system of the two particles as such, quite irrespectively of the relations in which these particles may stand to all other bodies in space.

Applying this fundamental principle to the *kinetic energy* of two particles in the same way as it has just been done in respect of the *potential energy,* we see that the *kinetic energy* must be taken as dependent upon the *first* part of the total *vis viva* belonging to the two particles—that is to say, upon their *relative* vis viva—and not upon the *second* part of the total *vis viva,* or that which the two particles possess in virtue of their revolution about one another in space or of the motion of their centre of gravity in space; for this latter part depends upon relations which the two particles do not of themselves directly present. For the two particles taken by themselves do not directly present any relation to space except their distance apart, from which no knowledge can be had of their rotation or of the motion of their centre of gravity in space.

Consequently, in what follows, by the *kinetic energy* of two particles is to be understood, not the total *vis viva* possessed by the two particles, but only their *relative* vis viva.

But it is easy to see that, in accordance with this, while a system of two electrical particles $e, e'$ receives from without an amount of kinetic energy $=a$, it really undergoes an alteration of its *potential energy* $=-a$; so that the *whole* energy of the system must always retain the same value not only when the two particles constitute a detached system, but also when they do not do so. For if we represent the *kinetic energy* communicated from without by

$$a=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv,$$

while the kinetic energy of the particles *before* the communication of this portion was

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot u_0u_0,$$

the kinetic energy existing *after* the communication is

$$\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot uu=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(u_0u_0+vv).$$

Consequently the *potential energy before the communication* is

$$-\frac{r}{\rho}\mathrm{V}=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-u_0u_0),$$

whereas the *potential energy after the communication* is

$$-\frac{r}{\rho}\mathrm{V}=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-uu)=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-u_0u_0)-\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv;$$

so that, in consequence of the communication from without of *kinetic energy* equal to $+a$, a change of *potential energy* has occurred which is represented by

$$-\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv=-a.$$

### 7. *Application to other Bodies.*

If we distinguish, in accordance with the last section, between the potential and the potential energy of two particles—that is to say, if we define

*Potential* as the amount of *work* which, in consequence of the mutual action of the two particles, is done during the transference of the particles from an infinite distance to the actual distance $r$ with the existing relative velocity $\frac{dr}{dt}$; and

*Potential energy* as that amount of *work*, taken *negatively*, which, in consequence of the mutual action of the two particles, is done during the transference of the particles *from the greater distance* $r = \infty$ *to the smaller distance* $r = \rho$ determined by the particles $e$, $e'$, their masses $\epsilon$, $\epsilon'$, and by the constant $c$, with the existing relative velocity $\frac{dr}{dt}$,—

the latter (that is to say, the *potential energy in the sense that has been indicated*) may be resolved into two parts, one of them equal and opposite to the *potential*, and therefore identical with the magnitude which has *hitherto* been *alone called potential energy*, but which, regarded henceforward as only a part of the potential energy, we may call the *free potential energy*; the remainder is the *second* part, which may be called the *latent potential energy*.

Hence the principle of the conservation of energy may be enunciated in the first place in the *earlier* wider sense as follows:—

For a *detached* system of two particles the sum of the *kinetic energy* and of the *free potential energy* is always the same.

For so long as no kinetic energy is either lost or communicated from without, every change in the free potential energy will be compensated by an equal and opposite change in the kinetic energy.

But the principle of the conservation of energy may also be enunciated, secondly, in the *narrower sense* as follows (potential energy and kinetic energy being understood in the sense that has just been defined):—

The *relative kinetic energy* of two particles, and the *total potential energy* which they possess along with this kinetic energy, together give always the same sum.

Upon this the following remarks may be made:—

(1) One particle regarded by itself can only possess *kinetic energy*.

(2) Two particles likewise possess in the first place kinetic energy, which is the sum of those which they possess when considered separately.

(3) This sum consists of a part A, which may be ascribed partly to the motion of their centre of gravity, and partly to their rotation about one another in space—and of another part B, which the particles possess relatively to each other when considered by themselves. This latter part, B, is called the *relative kinetic energy,* or *that belonging to the system formed by the two particles.*

(4) But in the *system of two particles* there is a something, in addition to its kinetic energy, which does not belong to the two particles taken separately, namely a greater or less *capacity for doing work* in virtue of the mutual action of the two particles upon each other. The *measure* of this capacity for doing work is termed the *potential energy of the system,* or the *relative potential energy of the two particles*; and that quantity of work serves as the *measure of this working-power* which is done in consequence of the mutual action of the two particles during their transference *from the smaller distance* $r=\rho$ *to the greater distance* $r=\infty$, where $\rho$ is determined by the particles themselves $e$, $e'$, by their masses $\epsilon$, $\epsilon'$, and by the constant $c$.

(5) The principle of the conservation of energy, however, when specially defined as above, is only applicable to two particles when their *potential* is of the same form as that of two electrical particles, namely

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right).$$

The potential of two ponderable masses $m$, $m'$, on the contrary, is

$$V=\frac{mm'}{r},$$

which (neglecting the sign) can be included under the above general form only if the value of the constant $c$ for ponderable masses is infinitely great. It is evident, however, that it would in reality suffice for the constant $c$ to have only a very great value instead of an infinite value, in order that there might not be any thing perceptibly inconsistent with the results of experiment. And, considering the extraordinarily high value which must be ascribed to the constant $c$ in the case of electrical particles, it does not seem at all necessary, for the avoidance of all sensible contradictions, to adopt any other value for ponderable bodies; consequently it must be permissible to represent the *potential* of two ponderable particles $m$, $m'$ by

$$V=\frac{mm'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right),$$

where the constant *c* retains *the same value as in the potential of two electrical particles.*

But even if it should hereafter result from more accurate experimental results that it is not permissible thus to ascribe the same value to the constant *c* in the case of ponderable particles, the possibility would always remain of assigning to the constant *c* a still greater value for ponderable particles; and this could easily be taken so great that any sensible disagreement with experiment would completely vanish.

[To be continued.]

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## II. *Further Notes on the Theory of the Tides.* *By the* Rev. T. K. Abbott, *Trinity College, Dublin**.

IN the demonstrations given in two previous papers in this Magazine (January 1870 and February 1871), we have supposed the water to be limited to an equatorial canal, the moon also being in the equator. It is desirable to consider what modifications will be introduced, first, by supposing the earth to be uniformly covered with water, and, secondly, by taking into account the moon's declination.

It will save repetition if we state once for all certain general principles which we shall have to employ. First, suppose an accelerating force acts alternately in opposite directions, the effect (measured by velocity) increases as long as the force acts in either direction, and therefore the velocity in that direction is greatest at the moment that the force changes its direction. Secondly, the velocity (diminishing under the counteraction of the new force) continues to be in the same direction until this counter force has undone all the work done in that direction by the previous force. When the circumstances are alike in both directions, this will be when the force has done half its work. This is precisely the case of the common pendulum. Thirdly, in the case before us, the water rises when the particles behind are moving faster than those before. The rate of rise is greatest when this difference is greatest; but as the effect is cumulative, the whole amount of the rise is greatest at the moment when the difference = 0, and is about to change to the opposite. Fourthly, as in 2, this difference ceases to increase (*i. e.* is greatest) when the force (or difference of forces) producing it ceases to act; but it is not reduced to 0 until the opposite force has done half its work. At this moment the accumulation is greatest. Fifthly, in the case which we are now considering, the effective force depends on the form of the surface, and *vice*

* Communicated by the Author.

XV. *Electrodynamic Measurements. By* Professor WILHELM WEBER.—Sixth Memoir, *relating specially to the Principle of the Conservation of Energy.*

[Concluded from p. 20.]

8. *On the Movement of two Electrical Particles in consequence of their action on each other.*

THE fundamental electrical law determines the action exerted by any given particle upon another under any circumstances. The simplest and most obvious application that can be made of this law, would seem to be to develope the laws of the motion of two particles which act mutually upon each other. Greater practical interest, however, attached to the determination, in the first place, of the laws of the distribution of electricity at rest upon conductors, and of the laws of the forces exerted by a current of electricity in a closed conductor, by reason of the current existing in another conductor, upon this latter conductor itself—as well as to the development of the laws of the (electromotive) forces exerted by closed currents (or by magnets) on the electricity in closed conductors—inasmuch as the results of these developments admitted of being directly tested and confirmed by experiment. But although this important practical interest is wanting to the development of the laws of motion of two particles subject only to their mutual action, many of its results cannot fail to merit attention in other respects.

The interest which belongs to these results relates indeed specially to the *molecular movements* of two particles, movements which are shut out from all direct experimental investigation, so that there is no authority for the application to them of the law that has been established, so far as it is regarded as an experimental law. Consequently the development of the laws of the *molecular movements* of two particles in accordance with the law that has been established must be considered only as an attempt to find a clue to the theory (which as yet we are entirely without) of these movements—a clue which by itself is certainly not sufficient, but is still in need of being supplemented in essential respects. For so long as the *molecular forces acting only at molecular distances,* which doubtless cooperate in the molecular movements, are not known and taken exact account of, the results that may be acquired cannot have any exact *quantitative* application, but only a *qualitative* value within certain limits, and can be of consequence only for a first *reconnaissance* of the territory.

9. *Motion of two Electrical Particles in the direction of the straight line which joins them.*

For two particles, $e$, $e'$, moving simply in consequence of their mutual action, we have, according to the fundamental laws of Section 4, by putting

$$\rho=2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc},\quad x=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2},\quad a=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot cc$$

and also giving a negative sign to U and V, so as to denote thereby the *potentials*,

$$V:U=2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}:r$$

$$-U+\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot cc;$$

and therefore

$$V=\frac{2}{r}\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}.\ U=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right).$$

If there is no *motion of rotation of the particles about each other* in space, $\frac{1}{\epsilon}\cdot\frac{dV}{dr}$ is the acceleration of the particle $e$ in the direction of $r$, and $\frac{1}{\epsilon'}\cdot\frac{dV}{dr}$ is the acceleration of the particle $e'$ in the opposite direction. Hence the *relative acceleration* of the two particles becomes

$$\frac{ddr}{dt^2}=\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{dV}{dr};$$

and from this, by integrating between the limits $r=r_0$ and $r=r$ ($r_0$ denoting the value of $r$ for the moment when $\frac{dr}{dt}=u=0$), since $\rho$ was made $=2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}$, we obtain

$$\frac{dr^2}{dt^2}=uu=\frac{r-r_0}{r-\rho}\cdot\frac{\rho}{r_0}\cdot cc.$$

$\frac{\rho}{r_0}$ has always a positive or negative value differing from nothing; for $\rho=2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}$ has a given finite although very small value, which is positive or negative according as $ee'$ is positive or negative; and $r_0=\dfrac{r}{r+\frac{uu}{cc}\cdot\frac{r-\rho}{\rho}}$ has also a positive or negative value differing from nothing, since the initial values of $r$ and $uu$, by which $r_0$ is to be determined, must be considered as *positive measurable quantities* to be determined by experiment.

When $\frac{\rho}{r_0}$ is positive because both numerator and denominator are positive, all the movements are confined to the distances outside the interval $\rho r_0$, and are divisible into *movements at a distance* and *molecular movements* which are separated from each other by the interval $\rho r_0$.

But if $\frac{\rho}{r_0}$ is positive because numerator and denominator are both negative, the movements extend to all possible distances, since the interval $\rho r_0$ then lies outside all possible distances.

When $\frac{\rho}{r_0}$ is negative, in which case the interval $\rho r_0$ lies partly outside and partly within the possible distances, all the movements are confined to the part of the interval $\rho r_0$ lying within possible distances; and if $\rho$ is positive and $r_0$ negative, they are *molecular movements.*

From this it follows, when $\rho$ and $r_0$ are positive, that, in the first place, no transition from *movements at a distance* to *molecular movements* takes place; secondly, that $uu$ always remains less than $cc$, if it was smaller at first; and thirdly, that when $uu$ is less than $cc$, $r$ and $r_0$ are (both at once) either greater or less than $\rho$.

If we keep merely to experience, some of these relative movements of the two particles may be left entirely out of account, for it is evident that infinitely great relative velocities are never met with in reality; on the contrary, $\frac{1}{cc}\cdot\frac{dr^2}{dt^2}$ is almost always to be considered a very small fraction.

This limitation, derived from the nature of things, is also tacitly assumed when $V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right)$ is taken as the *potential,* since this must be $=0$ for an infinitely great value of $r$. For if $\frac{dr^2}{dt^2}$ were infinitely great, the expression $\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right)$ might have a value differing from nothing even for infinitely great values of $r$.

But if the value of $\frac{dr^2}{dt^2}$ is never infinitely great, there must be a finite value which $\frac{dr^2}{dt^2}$ never exceeds. We may assume $cc$ as such a value.

Presupposing this limitation of the relative velocities, $r_0$ is always positive; and for every value of $r_0$ there exists only a single, always continuous series of corresponding values of $r$ and

$\frac{dr^2}{dt^2}$; and

when $\rho$ is positive and $r_0$ is $>\rho$,

the corresponding values of $r$ and $\frac{dr^2}{dt^2}$ extend from $r=r_0$ to $r=\infty$ and from $\frac{dr^2}{dt^2}=0$ to $\frac{dr^2}{dt^2}=\frac{\rho}{r_0}$. The movements are in this case *movements at a distance.*

If $\rho$ is positive and $r_0<\rho$, or if $\rho$ is negative,

the corresponding values extend from $r=r_0$ to $r=0$, and from $\frac{dr^2}{dt^2}=0$ to $\frac{dr^2}{dt^2}=cc$. In the first case, when $\rho$ is positive and $r_0<\rho$, and likewise in the second case, when $\rho$ is negative and $r_0<\rho$, the movements are *molecular movements*; but if, in the second case, $r_0$ is $>\rho$, the movements are partly *movements at a distance* and partly *molecular movements.*

Hence, with the above limitation of the movements, we obtain for two particles $e$, $e'$, moving solely in consequence of their reciprocal action, if there is *no motion of rotation of the particles about each other* in space, the following equation of motion, namely,

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\cdot\frac{\rho}{r_0},$$

in which $u$ is put $=\frac{dr}{dt}$, and where $\rho$ has a value that is given by the particles $e$, $e'$, their masses $\epsilon$, $\epsilon'$, and the constant $c$, and $r_0$ denotes a constant to be determined, according to this very equation, by the initial value of $r$ (which must be positive and not equal to $\rho$, but otherwise may be any thing whatever) and the initial value of $uu$ (which must be positive and less than $cc$, but otherwise may be any thing whatever).

10. *Two states of aggregation of a system of two particles of the same kind.*

For two like particles the value of $\rho$ is positive. And since, moreover, for every value of $r$ the relative velocity $u$ may have two equal but opposite values, the value of $r$ may, in accordance with the above equation $\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\cdot\frac{\rho}{r_0}$,

either *at first* decrease from $r=\infty$ to $r=r_0$, $u$ at the same time

increasing from $u=-c\sqrt{\frac{\rho}{r_0}}$ to $u=0$, and *afterwards*

$r$ may increase again from $r=r_0$ to $r=\infty$, $u$ at the same time increasing from $u=0$ to $u=+c\sqrt{\frac{\rho}{r_0}}$;

or $r$ may *at first* decrease from $r=r_0$ to $r=0$, $u$ at the same time decreasing from $u=0$ to $u=-c$, and then *afterwards* $r$ may increase from $r=0$ to $r=r_0$, $u$ at the same time decreasing from $u=+c$ to $u=0$.

It is easily seen that in the *first* case the motion is *not a reverting one*; for, after the distance $r$ has diminished from any given value to $r_0$, it increases again without limit; that is, it never decreases again. In the *latter* case, on the other hand, the motion is *reverting*, for the distance $r$ alternately diminishes from $r_0$ to 0 and increases again from 0 to $r_0$.

There seems indeed to be a sudden change in the value of the velocity $u$ from $-c$ to $+c$ at the moment when $r=0$; but no sudden change occurs in reality; for, when $r$ vanishes, $-c$ denotes the same velocity as $+c$ does when $r$ is increasing again from zero.

These two cases of motion are moreover distinguished from each other by the fact that *no transition* takes place from one to the other; for, according to the above equation, such a transition, in the case of the interval $\rho r_0$ or $r_0\rho$ could only occur by $u$ taking imaginary values.

Now upon this separateness of the two kinds of motion a distinction may be founded between *two states of aggregation of a system of two similar particles*—that is, between a state of aggregation in which the particles can only move at a distance from each other, and a state of aggregation in which they can take part only in molecular movements. A transition from the one state of aggregation to the other cannot take place so long as both particles move in consequence of their reciprocal action only.

It only remains to be noted further, that it has been here presupposed that the two particles, considered in space, possessed no motion except in the direction of $r$; but in the next section the opposite case will be considered.

11. *Motion of two Electrical Particles which move in space with different velocities, in directions at right angles to the straight line joining them.*

Let $\alpha$ denote the difference of the two velocities which two electrical particles $e$ and $e'$, at a distance $r$ from each other, possess in space in a direction perpendicular to the straight line $r$ which joins them; then $\frac{\alpha\alpha}{r}$ denotes the part of the relative acceleration $\frac{du}{dt}$ which depends upon $\alpha$.

If we deduct this part $\frac{\alpha\alpha}{r}$ from the total acceleration $\frac{du}{dt}$, the difference $\left(\frac{du}{dt}-\frac{\alpha\alpha}{r}\right)$ expresses that part of the relative acceleration of the two particles which results from the forces exerted by them upon each other. According to section 9 this latter part was $=\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{dV}{dr}$; and hence we obtain the following equation,

$$\frac{du}{dt}-\frac{\alpha\alpha}{r}=\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{dV}{dr}.$$

Multiplying this equation by $udt=dr$, we get

$$udu-\alpha\alpha\frac{dr}{r}=\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\cdot\frac{dV}{dr}dr;$$

and hence, by integrating from the instant at which $u=0$, the value of $r$ corresponding to this instant being denoted by $r_0$,

$$\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)(V-V_0)=\tfrac{1}{2}uu-\int_{r_0}^{r}\frac{\alpha\alpha}{r}dr,$$

in which $V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right)$ and $V_0=-\frac{ee'}{r_0}$, but where, in order to perform the last integration, $\alpha\alpha$ must be represented as a function of $r$.

Now $r.\alpha dt$ is the element of surface described by the line connecting the two repelling or attracting particles while they move about each other for the element of time $dt$; and for equal elements of time $dt$ this superficial element retains always the same value, whence $r\alpha dt=r_0\alpha_0 dt$. Introducing the resulting value

$$\alpha\alpha=r_0r_0\alpha_0\alpha_0\cdot\frac{1}{rr}$$

in the last member of the above equation, and carrying out the integration, we obtain the following equation,

$$2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}\left(\frac{r-r_0}{rr_0}+\frac{1}{r}\cdot\frac{uu}{cc}\right)=\frac{uu}{cc}+\frac{\alpha_0\alpha_0}{cc}\cdot\frac{r_0r_0-rr}{rr};$$

from which, by putting $2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}=\rho$, the equation of motion

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\cdot\frac{\alpha_0\alpha_0}{cc}\right)$$

is obtained. Putting this value of $\frac{uu}{cc}$ into the equation

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

we get

$$V=\frac{ee'}{r}\left(\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\cdot\frac{\alpha_0\alpha_0}{cc}\right)-1\right),$$

$$\frac{dV}{dr}=\frac{ee'}{r}\cdot\frac{r_0-\rho}{(r-\rho)^2}-\frac{ee'}{(r-\rho)^2}\left(1-\left(3-2\frac{\rho}{r}\right)\frac{r_0r_0}{rr}\right)\frac{\alpha_0\alpha_0}{cc}.$$

## 12.

According to the last section, there exists an equation between the relative velocity $u$ and the relative distance $r$ of two particles moving anyhow *in space* under the action of their reciprocal forces, namely the equation

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\frac{\alpha_0\alpha_0}{cc}\right),$$

in which $\rho$ denotes a constant that is *positive* for two *similar* particles, and *negative* for two *dissimilar* particles.

Now from this there follow results relative to the free motions of two particles *in space*, which move, under the influence of their own reciprocal action, with unequal velocities in a direction perpendicular to the straight line joining them, quite similar to those arrived at in relation to the motions considered in section 10 *in the direction of the straight line* r. There results, in fact, in this case also, a distinction between two states of aggregation for two *similar* particles—namely, a state of aggregation in which the two particles move in such a way as to return periodically into the same position relatively to each other, and a state of aggregation in which the two particles move so as to become always more and more distant from each other and never return to the same position. No transition from one state of aggregation to the other takes place so long as the two particles move only under the influence of their own reciprocal forces.

## 13.

A rotation of the two particles about each other implies the existence of a certain *attracting force* if the two particles are to remain at a constant distance from each other during this rotation; and this attracting force required for the rotation increases, for the same distance, according to the square of velocity of rotation. According to this, one would expect that, for two *similar* electrical particles at a distance $r_0<\rho$ (at which they attract each other), there would be always a *certain velocity of rotation* $\alpha_0$ for which the attracting force required by the rotation should be equal to the attracting force resulting from the reciprocal action of the two particles, so that the two particles rotating about each other would remain, for this velocity of rotation, at the same distance $r_0$. This, however, is not the case, since the attracting force resulting from the reciprocal action of the two particles depends not only upon the distance $r_0$, but also upon the velocity of rotation $\alpha_0$, and increases with the latter in such a manner that it always remains greater than the attracting force required by the rotation, so that with any such rotation there is always involved a mutual approach of the two particles.

It follows indeed easily that, in the case of two *similar* particles $e$ and $e'$, when $\rho$ has a *positive* value and $r=r_0$, and consequently $u=0$, there is no value of $\alpha_0$ for which $\frac{du}{dt}=0$, as must be the case if the two particles are to remain at an invariable distance $r_0$. For when $r=r_0$, it results from the equation at the end of section 11 that

$$\frac{dV}{dr}=\frac{ee'}{r_0(r_0-\rho)}\left(1+2\frac{\alpha_0\alpha_0}{cc}\right);$$

and from this it further follows, since

$$\frac{du}{dt}-\frac{\alpha\alpha}{r}=\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{dV}{dr}=\frac{\rho}{2}\frac{cc}{ee'}\cdot\frac{dV}{dr},$$

that

$$\frac{du}{dt}=\frac{1}{2}\frac{cc}{r_0-\rho}\left(\frac{\rho}{r_0}+2\frac{\alpha_0\alpha_0}{cc}\right),$$

whence $\frac{du}{dt}$ can be equal to nothing only when

$$\alpha_0\alpha_0=-\frac{1}{2}\frac{\rho}{r_0}cc,$$

which for a *positive* value of $\rho$ (that is, when $e$ and $e'$ are of the *same kind*) is impossible.

It follows further that, in the case of two *similar* particles, if $r=r_0$, $\frac{du}{dt}$ is either *positive* or *negative*, according as $r_0>\rho$ or $r_0<\rho$. Consequently the two particles separate always to a greater and greater distance from each other when $r=r_0>\rho$, and approach always nearer to each other when $r=r_0<\rho$, whatever value $\alpha_0$ may have.

## 14. *On the Time of Oscillation of an Electrical Atomic Pair.*

Two *similar* electrical particles at a distance $r_0<\rho$ from each other (at which their relative velocity $=0$) do not remain at this distance, but approach each other from $r=r_0$ to $r=0$ with a velocity which increases from $u=0$ to $u=\sqrt{\left(cc+\frac{r_0r_0\alpha_0\alpha_0}{\rho}\cdot\frac{1}{r}\right)}$—that is to say, becomes infinite, if the velocity of rotation $\alpha_0$ differed from nothing for the instant at which $r=r_0$. From this it follows that the interval of time $\Theta$ in which the two particles approach each other from the distance $r=r_0$ to $r=0$ has a finite value. The fact that for the instant at which $r$ becomes equal to 0 the value of the relative velocity of the two particles becomes

$$\sqrt{\left(cc+\frac{r_0r_0\alpha_0\alpha_0}{\rho}\cdot\frac{1}{r}\right)}=\pm\infty,$$

signifies here only that this relative velocity is to be henceforward taken as a velocity of separation $=+\infty$, whereas it was, up to this point, a veloctiy of approach $=-\infty$. This being premised, it easily follows that, in a second equal interval of time $\Theta$, the two particles will separate from each other again from the distance $r=0$ to the distance $r=r_0$. The interval of time $2\Theta$, in which the two particles approach each other with increasing velocity from the distance $r=r_0$ to $r=0$ and then separate again from the distance $r=0$ to $r=r_0$, may be called the *time of oscillation* of the *atomic pair* formed of the two electrical particles.

There still remains the problem of *determining the time of oscillation* $2\Theta$ *of such an atomic pair.*

This time of oscillation can be readily deduced from the equation

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r_0+r}{r}\cdot\frac{\alpha_0\alpha_0}{cc}\right),$$

if it be assumed that therein $r_0$ is not greater than $\rho$.

For if we *first* consider the limiting case in which $r_0=\rho$, it follows from the above equation that

$$uu=cc+\alpha_0\alpha_0+\rho\alpha_0\alpha_0\cdot\frac{1}{r};$$

and hence, putting $u=\frac{dr}{dt}$,

$$dt=-dr\sqrt{\frac{r}{\rho\alpha_0\alpha_0+(cc+\alpha_0\alpha_0)r}}.$$

From this we obtain, by integration,

$$\Theta=-\int_{\varrho}^{0}dr\sqrt{\frac{r}{\rho\alpha_0\alpha_0+(cc+\alpha_0\alpha^0)r}}.$$

Accordingly we get:—

$$\Theta = \frac{\rho}{cc+\alpha_0\alpha_0}\sqrt{(cc+2\alpha_0\alpha_0)}$$

$$-\frac{\rho\alpha_0\alpha_0}{(cc+\alpha_0\alpha_0)^{\frac{3}{2}}}\log\left(\sqrt{\left(1+\frac{cc}{\alpha_0\alpha_0}\right)}+\sqrt{\left(2+\frac{cc}{\alpha_0\alpha_0}\right)}\right);$$

or, for small values of $\frac{\alpha_0}{c}$,

$$\Theta = \frac{\rho}{c}\left(1-\frac{\alpha_0\alpha_0}{cc}\log\frac{2c}{\alpha_0}\right).$$

If we *next* confine ourselves to the consideration of *small oscillations* (that is to say, those for which $\frac{r_0}{\rho}$ is very small), it results from the above equation, when $r_0$ and $r$ are taken as vanishingly small compared with $\rho$, that

$$uu = \frac{r_0 r_0 \alpha_0\alpha_0}{\rho}\cdot\frac{1}{r}+cc-\left(\frac{cc}{r_0}+\frac{\alpha_0\alpha_0}{\rho}\right)r;$$

whence, putting $u=\frac{dr}{dt}$,

$$cdt = -dr\sqrt{\frac{r}{\frac{r_0 r_0\alpha_0\alpha_0}{\rho cc}+r-\left(\frac{1}{r_0}+\frac{\alpha_0\alpha_0}{\rho cc}\right)rr}},$$

which leads to an elliptic integral. For vanishing values of $\frac{\alpha_0}{c}$, we obtain

$$cdt = -dr\sqrt{\frac{1}{1-\frac{r}{r_0}}};$$

whence there comes, by integration,

$$\Theta = -\frac{1}{c}\int_{r_0}^{0}\frac{dr}{\sqrt{\left(1-\frac{r}{r_0}\right)}} = \frac{2r_0}{c}.$$

When, as has been assumed, $r$ is $<\rho$, $r_0$ may be called the amplitude of oscillation; and it follows that, for small values of $\frac{\alpha_0}{c}$ and for small amplitudes of oscillation, the time of oscillation $2\Theta$ of an electrical atomic pair is proportional to the amplitude of oscillation $r_0$. But the factor with which $r_0$ must be multiplied in order to give $2\Theta$, though a constant $=\frac{4}{c}$ for small amplitudes, diminishes for greater amplitudes, and becomes $=\frac{2}{c}$ for the amplitude $r=\rho$.

If we put $c = 439450 \, . \, 10^6 \, \frac{\text{millimetre}}{\text{second}}$, it follows from this last determination that the value of $\rho$ must lie approximately between $\frac{1}{4000}$ and $\frac{1}{8000}$ of a millimetre in order that these oscillations may be equal in rapidity to those of light.

The difference of the electrical particles $e$, $e'$ and of their masses $\epsilon$, $\epsilon'$ in the case of small values of $\frac{\alpha_0}{c}$ and small amplitudes, does not affect the oscillations at all; and in the case of greater amplitudes it affects them only so far as the value of $\rho$ depends upon it.

### 15. *Applicability to Chemical Atomic Groups.*

The distinction between two or more states of aggregation of bodies, according as they consist of simple atoms, or of atomic pairs, or of groups of more than two atoms, has acquired great importance in relation to *chemistry*. Now one, and now another state of aggregation occurs; and in many chemical processes a transition takes place from one to another; but the intermediate states which occur in the case of such transition cannot exist permanently, and those states of aggregation are consequently completely separate from each other as *permanent states*.

Now it is obvious that the *permanence* of some atomic conditions, which are distinguished as special states of aggregation, and the *want of permanence* in all other atomic conditions, may have its cause in the laws of the reciprocal action of atoms—that is, in the difference between the forces exerted upon each other by atoms according to the different relations in which they may stand towards each other. The cause of the permanence of some atomic states and of the want of this permanence in others has not hitherto been recognized in the laws of the reciprocal action of atoms; and it would doubtless be difficult to succeed in discovering this cause in such laws of reciprocal action as it has hitherto been attempted to establish and to assume for ponderable atoms.

The question consequently presents itself, whether the cause of the permanence of certain atomic states may not perhaps be found in such laws of mutual action as have here been established and assumed for electrical particles. Hence the movements of two electrical particles under the influence of the reciprocal action assigned to them, which have been followed out in the preceding sections, are of interest in connexion with this point also, since in them a cause has been really discovered upon which the existence of such permanent states of aggregation may be founded. And in relation to this it is to be specially observed that the same forces as those which determine the *states of aggregation of electricity* formed by simple atoms and by atomic pairs, may possibly also determine similar *states of aggregation of ponderable bodies*. For in the general distribution of electricity it must be assumed that an atom of electricity adheres to each ponderable atom. But if atoms of electricity adhere firmly to ponderable atoms, nothing will be altered in the relations of the electrical atoms except the *masses* which have to be moved by the forces acting on the electrical atoms. But in the preceding developments the *masses* are left undetermined, and are simply denoted by $\epsilon$ and $\epsilon'$; while the electrical particles themselves, to which the masses $\epsilon$ and $\epsilon'$ belong, are determined, without a knowledge of the values $\epsilon$ and $\epsilon'$, by the measurable quantities $e$ and $e'$. If now we take the values of $\epsilon$ and $\epsilon'$ so great as to include the masses of the ponderable atoms adhering to the electrical atoms, all the results that have been arrived at in reference first of all to *electrical atoms* merely, may also be applied to the ponderable atoms combined with the electrical atoms.

### 16. *On the state of Aggregation and Oscillation of two dissimilar Electrical Particles.*

In the case of two *dissimilar* electrical particles, the same equations hold good as in the case of two similar particles, namely those of section 11; that is to say,

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\cdot\frac{\alpha_0\alpha_0}{cc}\right),$$

$$V=\frac{ee'}{r}\left[\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\cdot\frac{\alpha_0\alpha_0}{cc}\right)-1\right],$$

$$\frac{dV}{dr}=\frac{ee'}{(r-\rho)^2}\left[\frac{r_0-\rho}{r_0}-\left(1-\frac{3r-2\rho}{r^3}\cdot r_0 r_0\right)\frac{\alpha_0\alpha_0}{cc}\right],$$

where $\rho=2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee}{cc}$; the only difference is, that when the particles are *dissimilar* $\rho$ has a *negative* value, because the product $ee'$ is *negative*. Besides these equations we have also $\alpha r=\alpha_0 r_0$ (since only such motions are considered as are made by two electrical particles under the action of their own reciprocal action), whence there follows, lastly, the equation

$$\frac{du}{dt}=\frac{1}{2}\frac{\rho cc}{ee'}\cdot\frac{dV}{dr}+\frac{r_0 r_0\alpha_0\alpha_0}{r_3}.$$

Hence it results that, as in the case of two similar electrical particles, when $r=r_0$,

$$\frac{dV}{dr}=\frac{ee'}{r_0(r_0-\rho)}\left(1+2\frac{\alpha_0\alpha_0}{cc}\right),$$

$$\frac{du}{dt}=\frac{1}{2}\frac{cc}{r_0-\rho}\left(\frac{\rho}{r_0}+2\frac{\alpha_0\alpha_0}{cc}\right);$$

and that, when also $\alpha_0=\sqrt{-\frac{\rho cc}{2r_0}}$ (which has now a real value, since $-\rho=-2\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{ee'}{cc}$ is positive for dissimilar particles), $\frac{du}{dt}=0$; according to which, when $r=r_0$ and $\alpha_0=\sqrt{-\frac{\rho cc}{2r_0}}$, the two particles in their rotation about each other *remain always at the same distance* $(=r_0)$ *apart*, a case which with two *similar* particles cannot occur at all.

It follows, however, further from the equation

$$\frac{uu}{cc}=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}+\frac{r+r_0}{r}\frac{\alpha_0\alpha_0}{cc}\right),$$

or, when we put $n$ for the constant value $-\frac{r_0r_0\alpha_0\alpha_0}{\rho cc}$, from the following equation,

$$-\frac{r-\rho}{\rho}\cdot\frac{uu}{cc}=\left(\frac{r}{r_0}-1\right)\cdot\left[n\left(\frac{1}{r_0}+\frac{1}{r}\right)-1\right],$$

that besides the value $r=r_0$, for which $u=0$ is given, there is in general also another value of $r$, namely $\frac{nr_0}{r_0-n}$, for which likewise $u=0$.

These two values of $r$, however, for which $u=0$, differ from each other sometimes to a greater and sometimes to a smaller extent, according to the value of $n$; and when $n=\frac{r_0}{2}$ (that is to say, when $\alpha_0=\sqrt{-\frac{\rho cc}{2r_0}}$), they coincide completely; and it is only when the two values of $r$ for which $u=0$ coincide thus that the previously mentioned case occurs, for which we have at the same time $u=0$ and $\frac{du}{dt}=0$; and consequently the two particles, while revolving round each other, remain at the same distance.

In all other cases in which the velocity $u=0$ (as, for example, when $r=2n-x$, where $x<n$) there is also a second value of $r$ —in this case $2n+\frac{nx}{n-x}$,—for which also the velocity $u=0$. $\frac{du}{dt}$

has then a positive value for $r=2n-x$, but diminishes and becomes equal to nothing between $r=2n-x$ and $r=2n+\frac{nx}{n-x}$; so that, for $r=2n+\frac{nx}{n-x}$, $\frac{du}{dt}$ has a negative value. It is evident from this that repulsion of the two particles takes place from $r=2n-x$ as far as the value of $r$ for which $\frac{du}{dt}=0$, and attraction from this point as far as $r=2n+\frac{nx}{n-x}$, and consequently that the two particles must always remain in *oscillatory motion relatively to each other within the indicated limits.*

### 17. *On Ampère's Molecular Currents.*

The molecular state of aggregation of two dissimilar electrical particles that has just been described, namely that in which the distance of the two particles alternately increases and diminishes between exactly defined limits and the path in which one particle moves about the other becomes a circular orbit at the two limits, is deserving of closer consideration, especially in those cases in which it is admissible to regard one of the particles as being at rest and the other particle as moving in a circle about the first.

The relation between the particles in respect of their participation in the motion depends upon the ratio of their masses $\epsilon$ and $\epsilon'$; and, according to section 15, the values of $\epsilon$ and $\epsilon'$ must include the masses of the ponderable atoms adhering to the electrical atoms. Let $e$ be the positive electrical particle, and let the negative particle be equal and opposite to it, and let it therefore be denoted by $-e$ (instead of by $e'$). Now let a ponderable atom adhere to the latter only, whereby its mass is so much increased that the mass of the positive particle becomes negligible in comparison. The particle $-e$ may then be regarded as being at rest, and the particle $+e$ alone as being in motion around the particle $-e$.

The two dissimilar particles, when in the molecular state of aggregation that has been described, consequently represent an *Ampèrian molecular current*; for it can be shown that they correspond completely to the assumptions which Ampère made in relation to the *molecular currents.*

In order to show this, let us develope the expression for the force which *the moving particle e* exerts upon any given element of a current. Let $ds'$ denote the length of the given element of current, $+e'ds'$ the positive, and $-e'ds'$ the negative electricity which it contains; and, lastly, let $u'$ denote the velocity of the positive particle $+e'ds'$, and $-u'$ the velocity of the negative particle $-e'ds'$. Also, let $r$ denote the distance of the element of current from the particle $e$, $u$ the velocity of the particle $e$, $x, y, z$ the coordinates of the particle $e$, $x', y', z'$ the coordinates of the element of current, $\Theta$ and $\Theta'$ the angles which the directions of $u$ and $u'$ make with $r$, and $\epsilon$ the angle between the directions of $u$ and $u'$.

Next, let the general expression for the repelling force of two electrical particles $e$ and $e'$ at the distance $r$, namely

$$\frac{ee'}{rr}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}+\frac{2r}{cc}\frac{ddr}{dt^2}\right),$$

be transformed as follows (see Beer, *Einleitung in die Elektrostatik, die Lehre vom Magnetismus und die Electrodynamik*, S. 251). First, let the equation

$$rr=(x-x')^2+(y-y')^2+(z-z')^2$$

be differentiated with respect to the time $t$; we then get

$$r\frac{dr}{dt}=(x-x')\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)+(y-y')\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)+(z-z')\left(\frac{dz}{dt}-\frac{dz'}{dt}\right),$$

or also

$$r\frac{dr}{dt}=r(u\cos\Theta-u'\cos\Theta').$$

By a second differentiation we get

$$\frac{dr^2}{dt^2}+r\frac{ddr}{dt^2}=\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)^2+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)^2+\left(\frac{dz}{dt}-\frac{dz'}{dt}\right)^2$$
$$+(x-x')\left(\frac{ddx}{dt^2}-\frac{ddx'}{dt^2}\right)+(y-y')\left(\frac{ddy}{dt^2}-\frac{ddy'}{dt^2}\right)$$
$$+(z-z')\left(\frac{ddz}{dt^2}-\frac{ddz'}{dt^2}\right),$$

wherein

$$\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)^2+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)^2+\left(\frac{dz}{dt}-\frac{dz'}{dt}\right)^2=u^2+u'^2-2uu'\cos\epsilon.$$

If now the acceleration of the one particle, whose components are $\frac{ddx}{dt^2}, \frac{ddy}{dt^2}, \frac{ddz}{dt^2}$, be denoted by N, and the angle which its direction makes with $r$ by $\nu$, and in like manner the acceleration of the other particle, whose components are $\frac{ddx'}{dt^2}, \frac{ddy'}{dt^2}, \frac{ddz'}{dt^2}$, by N′, and the angle which its direction makes with $r$ by $\nu'$, we obtain

$$\frac{x-x'}{r}\left(\frac{ddx}{dt^2}-\frac{ddx'}{dt^2}\right)+\frac{y-y'}{r}\left(\frac{ddy}{dt^2}-\frac{ddy'}{dt^2}\right)+\frac{z-z'}{r}\left(\frac{ddz}{dt^2}-\frac{ddz'}{dt^2}\right)$$
$$=N\cos\nu-N'\cos\nu'.$$

The substitution of these values gives

$$2\frac{dr^2}{dt^2}+2r\frac{ddr}{dt^2}=2(u^2+u'^2-2uu'\cos\epsilon)+2r(\text{N}\cos\nu-\text{N}'\cos\nu'),$$

$$3\frac{dr^2}{dt^2}\qquad=3(u\cos\Theta-u'\cos\Theta)^2.$$

The second equation subtracted from the first gives

$$-\frac{dr^2}{dt^2}+2r\frac{ddr}{dt^2}=2(u^2+u'^2-2uu'\cos\epsilon)-3(u\cos\Theta-u'\cos\Theta')^2$$
$$+2r(\text{N}\cos\nu-\text{N}'\cos\nu'),$$

whence the general expression for the repelling force of two electrical particles $e$ and $e'$ at the distance $r$, namely

$$\frac{ee}{rr}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2d}{cc}\frac{dr}{dt^2}\right),$$

is obtained in the following transformed shape,

$$=\frac{ee'}{ccrr}\left[cc+2(u^2+u'^2-2uu'\cos\epsilon)-3(u\cos\Theta-u'\cos\Theta)^2\right.$$
$$\left.+2r(\text{N}\cos\nu-\text{N}'\cos\nu')\right].$$

By substituting for the particle $e'$ the positive electricity in the given element of current, namely $+e'ds$, this expression gives the repelling force

$$\frac{ee'ds'}{ccrr}\left[cc+2(u^2+u'^2-2uu'\cos\epsilon)-3(u\cos\Theta-u'\cos\Theta')^2\right.$$
$$\left.+2r(\text{N}\cos\nu-\text{N}'\cos\nu')\right];$$

but by putting for the particle $e'$ the negative electricity in the given element of current, namely, $-e'ds'$, we obtain the repelling force

$$\frac{ee'ds'}{ccrr}\left[-cc-2(u^2+u'^2+2uu'\cos\epsilon)+3(u\cos\Theta+u'\cos\Theta)^2\right.$$
$$\left.-2r(\text{N}\cos\nu+\text{N}'\cos\nu')\right],$$

since in this case $\epsilon+\pi$, $\Theta'+\pi$, and $\nu'+\pi$ take the place of $\epsilon$, $\Theta'$, and $\nu'$; and these therefore give together the total repelling force between the moving particle $e$ and the whole element of current, namely

$$\frac{4ee'ds'}{ccrr}(3uu'\cos\Theta\cos\Theta'-2uu'\cos\epsilon-r\text{N}'\cos\nu').$$

The repelling force between the stationary particle $-e$ and the whole element of current, on the other hand, if $r$ denotes the distance of the stationary particle $-e$ from the given element of current, is

$$+\frac{4ee'ds'}{ccrr}\cdot rN'\cos\nu',$$

since in this case $u=0$. But the difference between the value given to $r$ here and that assigned to it previously (namely the distance from the particle $+e$, in motion about the particle $-e$, to the given element of current), may be regarded as a negligible fraction of $r$, so that we get, for the repelling force exerted by the moving particle $+e$ and stationary particle $-e$ together upon the element of current, the expression

$$\frac{4ee'ds'}{ccrr}(3\cos\Theta\cos\Theta'-2\cos\epsilon)\cdot uu'.$$

If we were to put in place of the moving electrical particle $+e$ a second element of current, the positive electricity of which, moving with the velocity $+\frac{1}{2}u$, was denoted by $+eds$, and whose negative electricity, moving with the velocity $-\frac{1}{2}u$, was denoted by $-eds$, we should obtain for the mutual repelling force of the two elements of current the value

$$=\frac{4eds\,.\,e'ds'}{ccrr}(3\cos\Theta\cos\Theta'-2\cos\epsilon')\,.\,uu';$$

that is to say, the same expression as before, if the electrical particle previously denoted by $+e$ (and moving with the velocity $u$) were taken as equal to the positive electricity contained in the second element of current, namely $+eds$ (moving with the velocity $\frac{1}{2}u$).

It follows from this that the rotatory motion of the electrical particle $+e$ about the stationary particle $-e$ replaces a circular double current, if the positive electricity contained in the latter is equal to $+e$ and moves in its circular orbit with half the velocity of the aforesaid electrical particle $+e$, and if also the negative electricity contained in the current is equal to $-e$ and moves with the same velocity as the positive electricity but in the opposite direction.

Hence it appears that an electrical particle $+e$ moving in a circle about the electrical particle $-e$ exerts upon all galvanic currents the same effects as those assumed by Ampère in the case of his molecular currents.

The molecular currents assumed by Ampère, however, differ essentially from all other galvanic currents in this respect, that, according to Ampère's assumption, they *continue* without electromotive force; whereas all other galvanic currents, in accordance with Ohm's law, are proportional to the electromotive force, and *cease* when the electromotive force vanishes. But it is evident that the electrical particle $+e$, spoken of above, must of itself, without electromotive force, continue indefinitely its rotatory motion about the particle $-e$, and therefore must correspond entirely with the molecular currents assumed by Ampère in this respect also.

We accordingly obtain in this way, as a deduction from the *laws of the molecular state of aggregation of two dissimilar electrical particles*, developed in the preceding section, a simple construction for the molecular currents assumed by Ampère without proof that their existence was possible.

18. *Movements of two dissimilar Particles in Space under the Action of an Electrical Segregating Force* (Scheidungskraft).

If $\pi+v$ denotes the angle which the direction of the electrical segregating force makes with $r$, and $a$ denotes the magnitude of the relative acceleration of the two particles depending upon the segregating force, $-a.\cos v$ and $a.\sin v$ are the components of $a$,—the *former* expressing the part of the relative acceleration $\frac{du}{dt}$ which is dependent on the segregating force, and the latter the part of $\frac{d\alpha}{dt}$ which depends on the same force, where $\alpha$ is the difference of the velocities of the two particles in a direction perpendicular to $r$. It is presupposed that the direction of the segregating force lies in the plane in which the two particles rotate about each other.

If now the *first* component, namely $-a.\cos v$, as the part of $\frac{du}{dt}$ which depends upon the *segregating force*, and also $\frac{\alpha\alpha}{r}$, as the part of $\frac{du}{dt}$ which depends upon the *velocity* $\alpha$, be deducted from the total acceleration $\frac{du}{dt}$, the difference

$$\left(\frac{du}{dt}+a.\cos v-\frac{\alpha\alpha}{r}\right)$$

denotes the part of the relative acceleration which results from the force which the two particles $e$ and $e'$ exert upon each other, namely

$$\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)\frac{dV}{dr}=\frac{\rho}{2}\frac{cc}{ee'}\cdot\frac{dV}{dr};$$

and hence the following equation is obtained:—

$$\frac{du}{dt}+a.\cos v-\frac{\alpha\alpha}{r}=\frac{\rho}{2}\frac{cc}{ee'}\cdot\frac{dV}{dr}$$

If we deduct the *last* component, namely $a.\sin v$, as the part of the acceleration $\frac{d\alpha}{dt}$ which depends upon the *segregating force,* from the total value $\frac{d\alpha}{dt}$, the difference $\left(\frac{d\alpha}{dt} - a \sin v\right)$ gives that part of the total acceleration $\frac{d\alpha}{dt}$ which results from *the existing motion under the sole influence of the forces exerted upon each other by the two particles.* But, under the sole influence of the attracting or repelling forces exerted upon each other by the two particles, the element of surface $\alpha r dt$, described in a given element of time $dt$, would have a constant value, or we should have $\alpha \frac{dr}{dt} + r \frac{d\alpha}{dt} = 0$; hence the resulting part of the acceleration $\frac{d\alpha}{dt}$ becomes

$$-\frac{\alpha}{r} \frac{dr}{dt}.$$

By equating this part with the above difference, we get the equation

$$\frac{d\alpha}{dt} - a \sin v = -\frac{\alpha}{r} \frac{dr}{dt}.$$

Besides these, we have, as is self-evident, a third equation,

$$dv = \frac{\alpha dt}{r}.$$

Accordingly, for the four variable magnitudes $r$, $u$, $\alpha$, $v$, there are the following three equations:—

$$a \cos v - \frac{\alpha\alpha}{r} = \frac{\rho cc}{2ee'} \cdot \frac{dV}{dr} - \frac{du}{dt}, \quad \ldots \quad (1)$$

$$a \sin v - \frac{\alpha dr}{r dt} = \frac{d\alpha}{dt}, \quad \ldots \quad (2)$$

$$dv = \frac{\alpha dt}{r}. \quad \ldots \quad (3)$$

Multiplying equation (1) by $dr = u dt$, and equation (2) by $r dv = \alpha dt$, we obtain

$$a \cos v \,.\, dr - \frac{\alpha\alpha dr}{r} = \frac{\rho cc}{2ee'} \cdot \frac{dV}{dr} dr - u du, \quad \ldots \quad (4)$$

$$a r \sin v \,.\, dv - \frac{\alpha\alpha dr}{r} = \alpha d\alpha. \quad \ldots \quad (5)$$

The difference of these two equations gives

$$a \,.\, d(r \cos v) = \frac{\rho cc}{2ee'} \cdot \frac{dV}{dr} dr - \alpha d\alpha - u du. \quad \ldots \quad (6)$$

We also get from (2) and (3),

$$-2ar^3 . d(\cos v) = d(\alpha^2 r^2). \quad . \quad . \quad . \quad . \quad (7)$$

The integration of the differential equation (6) gives, after multiplying by 2 and putting $V = \frac{ee'}{r}\left(\frac{uu}{cc} - 1\right)$,

$$2ar\cos v = \frac{\rho cc}{r}\left(\frac{uu}{cc} - 1\right) - \alpha\alpha - uu + \text{const.}; \quad . \quad . \quad (8)$$

and from this, since $r = r_0$, $\alpha = \alpha_0$, and $\cos v = -1$ when $u = 0$, comes

$$-2ar_0 = -\frac{\rho cc}{r_0} - \alpha_0\alpha_0 + \text{const.} \quad . \quad . \quad . \quad (9)$$

Equation (9), subtracted from equation (8), gives

$$2ar\cos v + 2ar_0 = \left(\frac{\rho}{r} - 1\right)uu + \rho cc\left(\frac{1}{r_0} - \frac{1}{r}\right) - \alpha\alpha + \alpha_0\alpha_0. \quad (10)$$

By integrating the differential equation (7) we obtain, after dividing by $r^3$,

$$-2a\cos v = \frac{\alpha\alpha}{r} + 3\int\frac{\alpha\alpha dr}{rr},$$

or, multiplying by $r$,

$$-2ar\cos v = \alpha\alpha + 3r\int\frac{\alpha\alpha dr}{rr}, \quad . \quad . \quad . \quad (11)$$

and hence, for the sum of (10) and (11),

$$2ar_0 = \left(\frac{\rho}{r} - 1\right)uu + \rho cc\left(\frac{1}{r_0} - \frac{1}{r}\right) + \alpha_0\alpha_0 + 3r\int\frac{\alpha\alpha dr}{rr},$$

and therefore

$$uu = \frac{1}{r-\rho}\left(\rho cc\left(\frac{r}{r_0} - 1\right) + r\alpha_0\alpha_0 + 3rr\int\frac{\alpha\alpha dr}{rr} - 2ar_0 r\right). \quad (12)$$

From equation (3) there follows further, since $dr = udt$,

$$dv = \frac{\alpha}{u}\frac{dr}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and since, by equation (7),

$$d(\cos v) = -\frac{d(\alpha^2 r^2)}{2ar^3},$$

and by equation (11),

$$\cos v = -\frac{1}{2a}\left(\frac{\alpha\alpha}{r} + 3\int\frac{\alpha^2 dr}{r^2}\right),$$

we get, by substituting these values in the identical equation

$$dv = -\frac{d(\cos v)}{\sqrt{(1 - \cos v^2)}},$$

according to equation (13),

$$\frac{\alpha}{u}\frac{dr}{r}=\frac{\dfrac{d(\alpha^2r^2)}{2ar^3}}{\sqrt{\left(1-\frac{1}{4a^2}\left(\frac{\alpha^2}{r}+3\int\frac{\alpha^2dr}{r^2}\right)^2\right)}};$$

and from this and equation (12),

$$uu=\left(\frac{\alpha r^2dr}{d(\alpha^2r^2)}\right)^2\cdot\left(4a^2-\left(\frac{\alpha^2}{r}+3\int\frac{\alpha^2dr}{r^2}\right)^2\right)$$

$$=\frac{1}{r-\rho}\left(\frac{r-r_0}{r_0}\rho c^2+r(\alpha_0^2-2ar_0)+\rho r^2\int\frac{\alpha^2dr}{r^2}\right), \quad (14)$$

or the following equation for the two variables $r$ and $\alpha$ :—

$$4a^2=\left(\frac{\alpha^2}{r}+3\int\frac{\alpha^2dr}{r^2}\right)^2$$

$$+\frac{4}{r-\rho}\left(\frac{d(\alpha r)}{dr}\right)^2\cdot\left(\frac{r-r_0}{r_0}\cdot\frac{\rho c^2}{r^2}+\frac{\alpha_0^2-2ar_0}{r}+3\int\frac{\alpha^2dr}{r^2}\right)^*. \quad (15)$$

If we now confine ourselves to small values of $a$, for which $\alpha r$ is not, indeed, constant, as it is for $a=0$, according to section 11, but for which it differs only little from a constant value $\alpha_0r_0=n$, we may put

$$\alpha r=n(1+\epsilon), \quad \ldots \ldots \quad (16)$$

where $\epsilon$ has always a very small value. It then follows from this that

$$\frac{\alpha^2}{r}=(1+2\epsilon)\frac{n^2}{r^3}, \quad \ldots \ldots \quad (17)$$

$$\frac{d(\alpha r)}{dr}=n\frac{d\epsilon}{dr}. \quad \ldots \ldots \quad (18)$$

Further, by (11) and (17),

$$\int\frac{d\epsilon}{r^3}=-\frac{a}{n^2}\cos v,$$

* If the segregating force $a$ vanish, $\alpha r$ must, according to section 11, assume a constant value. But for a constant value of $\alpha r$ and for $a=0$, equation (15) reduces itself to

$$0=\frac{\alpha^2}{r}+3\int\frac{\alpha^2dr}{r^2};$$

and this, divided by the constant value $\alpha^2r^2$, gives the identical equation

$$0=\frac{1}{r^3}+3\int\frac{dr}{r^4},$$

in accordance with section 11.

or

$$d\epsilon = \frac{a}{n^2} r^3 \sin v dv \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

from (18) and (19),

$$\frac{d(\alpha r)}{dr} = \frac{a}{n} r^3 \sin v \, . \, \frac{dv}{dr} \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and from (17) and (19),

$$\frac{\alpha^2}{r} = \frac{n^2}{r^3} + \frac{2a}{r^3} \int r^3 \sin v dv . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If we now substitute the values of $\frac{d(\alpha r)}{dr}$ and $\frac{\alpha^2}{r}$ given by (20) and (21) in the following equation resulting from (11) and (15), namely

$$a^2 \sin v^2$$

$$= \frac{1}{r-\rho} \cdot \left(\frac{d(\alpha r)}{dr}\right)^2 \cdot \left(\frac{r-r_0}{r_0} \cdot \frac{\rho c^2}{r^2} + \frac{\alpha_0^2 - 2ar_0}{r} - \frac{\alpha^2}{r} - 2a\cos v\right), (22)$$

we obtain, by again putting for $n$ its value $\alpha_0 r_0$, the following equation between $r$ and $v$, namely

$$\frac{\alpha_0^2 r_0^2}{r^4 c^2} \cdot \frac{dr^2}{dv^2} = \frac{r-r_0}{r-\rho} \left(\frac{\rho}{r_0} + \frac{r+r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right)$$

$$- \frac{2a}{(r-\rho)c^2} \left(r_0 r + \frac{3}{r} \int r^2 \cos v dr\right)^* . \quad . \quad (23)$$

By differentiating this equation, after multiplying it by $r(r-\rho)$, we obtain

$$\frac{d}{dr}\left((r-\rho)\frac{\alpha_0^2 r_0^2}{r^3 c^2} \cdot \frac{dr^2}{dv^2}\right) = \frac{\rho r}{r_0} + (r+r_0)\left(\frac{\alpha_0^2}{c^2} + (r-r_0)\left(\frac{\rho}{r_0} + \frac{\alpha_0^2}{c^2}\right)\right.$$

$$\left. - \frac{2a}{c^2} (2r_0 r + 3r^2 \cos v)\right).$$

* From the above equation, since $\frac{r}{\alpha} u$ may be substituted for $\frac{dr}{dv}$, we obtain

$$\frac{\alpha_0^2 r_0^2}{\alpha^2 r^2} \cdot \frac{uu}{cc} = \frac{r-r_0}{r-\rho} \left(\frac{\rho}{r_0} + \frac{r+r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right) - \frac{2a}{(r-\rho)c^2}\left(r_0 r + \frac{3}{r}\int r^2 \cos v dr\right),$$

which, when the segregating force $a$ vanishes, and therefore, according to section 11, $\alpha r = \alpha_0 r_0$, passes over into the equation

$$\frac{uu}{cc} = \frac{r-r_0}{r-\rho} \left(\frac{\rho}{r_0} + \frac{r+r_0}{r} \cdot \frac{\alpha_0^2}{c^2}\right) -$$

that is to say, into the same equation that was arrived at already for this case in section 11.

If we here put, to consider a special case,

$$\rho = -\frac{2r_0}{cc}(\alpha_0^2 + ar_0)$$

(that is to say, the case in which, for $a=0$, the two particles remain, according to section 16, at the same distance during their rotation), we obtain

$$\frac{d}{dr}\left((r-\rho)\frac{\alpha_0^2 r_0^2}{r^3 c^2}\cdot\frac{dr^2}{dv^2}\right) = -\frac{2(r-r_0)}{cc}(\alpha_0^2 + ar_0) - \frac{6ar}{cc}(r_0 + r\cos v),$$

which becomes $=0$, *first*, when $u=0$ and consequently $r=r_0$, $\alpha=\alpha_0$, and $\cos v = -1$, and, *secondly*, when

$$r_0 - r = \frac{3ar(r_0 + r\cos v)}{\alpha_0^2 + ar_0},$$

a case which occurs for small values of $a$, if $\cos v = +1$ and so $r = r_0 - \frac{6ar_0^2}{\alpha_0^2}$ approximately.

Hence it follows that, just as, according to section 16, one of two dissimilar electrical particles, for which $\rho = -2r_0\frac{\alpha_0\alpha_0}{cc}$, could move round the other in a circular orbit when *not acted on by segregating force*, so also when two dissimilar electrical particles, for which $\rho = -2r_0\left(\frac{\alpha_0\alpha_0}{cc} + ar_0\right)$, *are acted on by a segregating force* ($=a$), one of them can revolve about the other in a closed orbit, though the orbit is not circular. The distance between the particles varies, in fact, according as the moving particle lies before or behind the central particle considered relatively to the direction of the segregating force, being in the latter case $=r_0$, and in the former case $=r_0 - 6\frac{r_0 r_0}{\alpha_0\alpha_0}a$.

Such an eccentrical position of the one particle in the plane of the orbit described (under the influence of a segregating force) by the other particle about this one, may be compared to the separation of electric fluids at rest under the influence of a similar segregating force; but the remarkable difference presents itself that the separation takes place in opposite directions in the two cases.

It follows from this, that in all conductors that have been charged in the usual way under the influence of a force of electrical segregation, the electricity cannot be contained only in the state of aggregation corresponding to Ampère's molecular currents, since in that case the resulting segregation would take place in the opposite direction to that which actually does occur. But even if all the electricity in such a conductor existed in the form of Ampèrian molecular currents before the action of the segregating force began, there must have been amongst these molecular currents some which could not persist under the action of the segregating force (one particle continuing to revolve in a closed orbit round the other), and were accordingly broken up, the two particles separating more and more from each other until they arrived at the boundary of the conductor. Under the influence of the force of segregation, the positive and negative particles of the broken molecular currents could remain at rest only when distributed in a particular way on the surface of the conductor; but when the force of segregation ceased to act, they would enter into motion again until they had again united themselves two by two into Ampèrian molecular currents.

### 19. *Electrical Currents in Conductors.*

If all the electricity in conductors were contained in them (before a segregating force began to act) in the state of aggregation corresponding to Ampèrian molecular currents, which, however, were incapable of persisting under the action of a segregating force, but were broken up, so that the two dissimilar electrical particles, which were revolving about each other, separated further and further from each other, until their paths finally approached asymptotically the direction of the segregating force, dissimilar electrical particles derived from different molecular currents would encounter each other before they could reach the boundaries of the conductor, and would form with each other new molecular currents. These newly formed molecular currents would then in their turn be broken up, and the particles constituting them would again separate further and further from each other in paths asymptotically approaching the direction of the segregating force, and so on.

Thus there would arise a current of electricity in the conductor in the direction of the segregating force. If the conductor had the shape of a uniform ring, and if the segregating force had the same intensity in every separate element of length of the ring and acted in the direction of the element, a constant circular current would be produced in the ring, and the laws of motion of electrical particles under the action of a force of electrical segregation, developed in the previous section, would form the basis of the theory of these constant electrical currents in closed conductors.

Here it is evident that, during the existence of this current, *work* would be done by each particle, since it moves forward under the action of the segregating force in the direction of this force. And since all the other forces which act upon such such a particle in a conductor must together balance each other, this work will make its appearance as an equivalent increase of the *vis viva* of the particle; whence it follows that the *vis viva* of all the Ampèrian molecular currents contained in the conductor must, while the current traverses the conductor, increase; that is to say, the square of the velocity with which the particles in the Ampèrian molecular current revolve about one another must increase proportionally to the force of segregation (*electromotive force*), and proportionally to the distance through which this force acts in its own direction (or to *the strength of the current*). If the ratio of the *electromotive force* to the *strength of the current* be called *resistance*, we may say instead of the above that the *vis viva* of all the molecular currents contained in the conductor increases, during the passage of the current, proportionally to the *resistance*, and proportionally to the *square of the strength of the current*.

This increase of *kinetic energy* of the electrical particles contained in a conductor while a current traverses it, follows therefore as a necessary consequence of the action of the electromotive force upon the particles, while these particles, as the result of the current, move onward in the direction of this force.

This theoretical conclusion receives, not indeed a direct, but an indirect confirmation from experiment, inasmuch as an increase of *thermal energy* is *observed* in the conductor while a current traverses it. And this *observed* increase of the *thermal energy* in the conductor is equal to the *calculated* increase of the *kinetic energy* of the electrical particles in the Ampèrian molecular currents of the conductor.

Now the *thermal energy* of a body is a *kinetic energy* resulting from movements in the *interior of the body*, which are therefore inaccessible to direct observation. In like manner, the *kinetic energy* belonging to the electrical particles in the Ampèrian molecular currents in a conductor is a kinetic energy which results from movements taking place in the *interior of the conductor*, and therefore inaccessible to direct observation.

But notwithstanding this agreement, the *thermal energy* of a body and this kinetic energy of the electrical particles in the Ampèrian currents contained in the same body might possibly be altogether different as to their essential nature. For it is possible that the *thermal energy* might be energy resulting from the motion of quite other particles than those of electricity, and the motion of these other particles might be of quite a different kind from those of the particles in Ampèrian currents.

In order to explain the identity of the increase of the energy of the Ampèrian molecular currents, as determined above, with the increase of thermal energy found by observation, it would then be absolutely necessary, *according to the principle of the conservation of energy*, that a *transference* should take place of the kinetic energy of the electrical particles in the Ampèrian currents to the other particles whose motion constitutes heat. And indeed it would be needful that *all* the kinetic energy produced by the current in the electrical particles of the Ampèrian currents should be *completely* transferred to these other particles at each instant.

But apart from the consideration that it is impossible to conceive how such a *complete* transference could take place, it is self-evident that any even partial transference of the kinetic energy of Ampèrian molecular currents to other particles is contradictory of the *permanence* which belongs to the essential nature of Ampèrian currents. If such a transference of kinetic energy from electrical particles in molecular currents to other particles were really to occur, it would simply prove that the molecular currents formed by these particles were not *Ampèrian molecular currents*, since they would not possess the permanence wherein the essence of Ampèrian molecular currents consists.

Hence it follows as a consequence that, if in conductors all the electrical particles exist in the state of aggregation corresponding to Ampèrian molecular currents, the observed increase in the *thermal energy* of a conductor, during the passage of a current through it, must result *immediately* from the increase of the *kinetic energy* of the electrical particles constituting the Ampèrian currents; that is to say, the *thermal energy* imparted to the conductor by the current must be *kinetic energy* due to motions in the interior of the conductor, and must in fact consist in *an increase in the strength of the Ampèrian currents formed by the electrical particles in the conductor.*

Reference may also be made, in connexion with the *identity of thermal energy and the kinetic energy of Ampèrian molecular currents*, to what is said respecting "the Transformation of the work of the current into Heat," in the 10th volume of the *Abhandlungen der K. Ges. d. Wiss. zu Göttingen* (1862), in the 33rd section of the memoir entitled "*Zur Galvanometrie.*"

## 20. *On Thermomagnetism.*

The following remark readily connects itself with the hypothesis of the previous section, that the electricity in conductors exists in the state of aggregation corresponding to Ampèrian molecular currents—and with the consequent identity of the *thermal energy* of the conductor and the *kinetic energy* of the Ampèrian currents in the conductor—namely, that *equality of temperature* in two conductors must depend upon certain relations between the strength and character of the Ampèrian currents in the two conductors, but that, along with the relation needed for this equality of temperature, the following difference may exist between the currents of the two conductors, namely:—that *greater masses* of electricity may move with *smaller velocity* in the Ampèrian currents of the one conductor, and *smaller masses* of electricity with *greater velocity* in those of the other conductor.

Let now a ring be conceived, formed of two such dissimilar conductors, through which a constant current passes, so that in the same time an equal quantity of electricity passes through every section of the ring; then it is evident that equal quantities of electricity must also traverse the two sections which bound the *first layer of the second conductor*. But the electricity which traverses the first section comes from the *first conductor*, in the molecular currents of which large masses of electricity move with small velocity. Hence, in consequence of this smaller velocity, this electricity which penetrates into the *first layer of the second conductor* possesses less *vis viva*. The electricity which passes through the second section comes from the above-mentioned first layer of the second conductor itself, where a smaller mass of electricity moves in the Ampèrian currents with a greater velocity, and therefore it possesses, in consequence of this greater velocity, a greater *vis viva*. It follows from this, that, as a consequence of the current, this *first layer of the second conductor* gives up more *vis viva* to the following layer of the second conductor than it receives from the last layer of the first conductor. Consequently a diminution takes place in the kinetic energy of the Ampèrian currents of this layer, or, in other words, a *diminution of the thermal energy or temperature*.

The opposite condition is found on considering the two sections which bound the *first layer of the first conductor*. The electricity which passes through the first section into this layer comes out of the end of the *second* conductor with a greater velocity; and that which passes out of this layer through the second section, leaves this section with a smaller velocity; whence it follows that, as a consequence of the current, the *first layer of the first conductor* gives up less *vis viva* to the following layer of the same conductor than it receives from the last layer of the second conductor; and thus an increase takes place in the kinetic energy of the Ampèrian currents of this layer, or, in other words, an *increase of the thermal energy or temperature*.

It will be seen that a foundation is here presented for the doctrine of *thermomagnetism*, and in particular for Peltier's fundamental experiment, although it would lead us too far to pursue it further here.

It may suffice merely to add here a similar remark in relation to Seebeck's fundamental thermomagnetic experiment. In a body which possesses the same temperature in all its parts, the heat is supposed to be in a state of *mobile equilibrium*; or we speak, with Fourier, of a *reciprocal radiation* of the particles of the body, by virtue of which each particle parts with just as much heat to the surrounding particles as it receives from them. Now, if heat consists in Ampèrian molecular currents, which, however, are broken up by the positive and negative particles separating from each other until they encounter other particles, with which they form new molecular currents, equilibrium of temperature must consist in this, that the *vis viva* of the electrical particles which leave any part of the body is equal to the *vis viva* of the electrical particles which enter this part of the body.

Let us now consider the surface of contact of two conductors which differ from each other only by greater masses of electricity moving with smaller velocity in the Ampèrian currents of one, and smaller masses moving with greater velocity in those of the other. Then, when both the conductors are at the same temperature, the *vis viva* of the electrical particles which pass from the first conductor into the second must be equal to the *vis viva* of the electrical particles that pass from the second conductor into the first; but the *mass* of the electrical particles which pass from the first conductor into the second would be greater than the *mass* of the electrical particles which pass out of the second conductor into the first. But from this (if the electricity which passes over is always positive, while the negative electricity remains behind in the conductor, to the particles of which it adheres) there would result a *difference of electrical charge on the two sides of the surface of contact*; that is to say, there would result *an electromotive force* at this surface of contact; for the electromotive force of a surface of contact is a force whereby a difference of electrical charge is produced at the two sides of the surface of contact.

If now the two conductors are of such a nature that this difference of charge at the two sides of their surface of contact is not always the same, but is *greater or less according to variations of temperature,* there would follow the production of a current in a ring formed of these two conductors, if different temperatures were to exist at the two surfaces of contact of the conductors.

### 21. *Helmholtz on the contradiction between the Law of Electrical Force and the Law of the Conservation of Force.*

In his memoir, "Ueber die Bewegungsgleichungen der Elektricität für ruhende leitende Körper," in the *Journal für die reine und angewandte Mathematik* (vol. lxxii. pp. 7 and 8), Helmholtz deduces from the law of electrical force the equation of motion of two electrical particles for motions in the direction of the distance $r$ of the two particles, namely

$$\frac{1}{cc} \cdot \frac{dr^2}{dt^2} = \frac{C - \frac{ee'}{r}}{\frac{1}{2}mcc - \frac{ee'}{r}},$$

or, putting $C = \frac{ee'}{r_0}$ and $\frac{2ee'}{mcc} = \rho$, the equation

$$\frac{1}{cc} \frac{dr^2}{dt^2} = \frac{r - r_0}{r - \rho} \cdot \frac{\rho}{r_0};$$

that is to say, the same equation as was arrived at in section 9.

If $\frac{ee'}{r} > \frac{1}{2}mcc > C$—that is, if $\frac{\rho}{r} > 1 > \frac{\rho}{r_0}$, we have $\frac{dr^2}{dt^2}$ positive and greater than $cc$, and $\frac{dr}{dt}$ is therefore real. If the latter is also positive, $r$ will increase until $\frac{ee'}{r} = \frac{1}{2}mcc$, that is till $r = \rho$, and then $\frac{dr}{dt}$ becomes *infinitely great.*

The same will happen if, to begin with, $C > \frac{1}{2}mcc > \frac{ee'}{r}$; that is, if $\frac{\rho}{r_0} > 1 > \frac{\rho}{r}$, and $\frac{dr}{dt}$ is negative.

These consequences are, according to Helmholtz, in contradiction with the law of the conservation of force.

Now it may be remarked hereupon, in the first place, that two electrical particles are here assumed which begin to move with a *finite* velocity certainly, but one which is greater than the velocity $c$—greater, that is, than $439450 \cdot 10^6 \frac{\text{millimetre}}{\text{second}}$. The case of two bodies moving relatively to each other with such a velocity is nowhere recognizable in nature. In all practical cases we are accustomed rather to treat $\frac{1}{cc} \frac{dr^2}{dt^2}$ as a very small fraction; and this deserves notice.

For, according to Helmholtz (*loc. cit.* p. 7), a law is in contradiction with the law of the *conservation of force* if two particles, moving in accordance with it and beginning with a *finite* velocity, attain, within a finite distance of each other, *infinite* vis viva, and so are able to do an infinitely great amount of work.

The principle seems to be here announced that, according to the law of the conservation of force, two particles cannot, under any circumstances, possess infinite *vis viva.*

For the above assertion may evidently be inverted, and we may say a law is in contradiction with the law of the conservation of force, if two particles, moving in accordance with it and beginning with *infinite* velocity, attain, at a finite distance from each other, finite *vis viva,* and thus suffer an infinitely great diminution of the work which they are able to perform.

The two particles must therefore always retain an infinite velocity; for if they have not lost it in any finite distance, however great, they would, in accordance with the nature of potential, never lose it even at greater distances. But bodies which always move relatively to each other with an infinite velocity are excluded from the region of our inquiries.

But if two particles never possess more than finite *vis viva,* there must be a finite limiting value of *vis viva* which they never exceed. It is consequently possible that this limiting value for two electrical particles $e$ and $e'$ may be $=\frac{ee'}{\rho}$; that is, that the square of the velocity, with which the two particles move relatively to each other, may not exceed $cc$.

The contradiction urged by Helmholtz would, according to this, lie not in the law, but in his assumption, according to which the two particles began to move with a velocity the square of which, namely $\frac{dr^2}{dt^2}$, was $>cc$.

If such a determination of the limiting value of *vis viva* is assumed in connexion with the *law of the conservation of force* according to Helmholtz, it may equally well be assumed in connexion with the *fundamental law of electrical action* (see section 4); that is, the *work* denoted there by U, as well as the *vis viva* denoted by $x$ $\left(\text{in the law } U+x=\frac{ee'}{\rho}\right)$, may *both* be regarded as being *by their nature positive quantities.*

In the second place, it may be remarked that, though the two electrical particles do attain infinite *vis viva* at a finite distance from each other, this finite distance is $\rho=\frac{2ee'}{cc}\left(\frac{1}{\epsilon}+\frac{1}{\epsilon'}\right)$, which, according to our measures, is an *undefinable small distance,* for the same reasons that the electrical masses $\epsilon$ and $\epsilon'$ are themselves undefinable according to our measures. This distance was consequently denominated in section 9 a molecular distance.

The *theory of molecular motions* requires in any case a special development, which as yet is wanting throughout. But as long as such a theory remains excluded from mechanical investigations, any doubts as to *physical admissibility* in relation to *molecular motions* are without foundation.

It may be remarked, in the third place, that the same objection, namely that two particles, which begin with finite velocity, attain infinite *vis viva* at a finite distance from each other, applies also to the law of gravitation, if it is assumed that the masses of ponderable particles are *concentrated in points*. But if this objection is got rid of, in the case of the law of gravitation, by assuming that the masses even of the smallest particles *occupy space*, we must make the same assumption in relation to electrical particles, in which case it results that only a vanishingly small part of such a particle arrives at a given instant at the distance $\rho$; another vanishingly small part, which arrived at the distance $\rho$ at the previous instant, will have exchanged its infinitely great velocity of approach for an infinitely great velocity of separation. But if these vanishing parts of the smallest particles are solidly connected together, there cannot be any question of such infinite velocities at all.

Even cosmical masses may begin their movements under physically admissible conditions, and, by continuing to move according to the law of gravitation, may come into physically inadmissible conditions, which can be avoided only through the cooperation of *molecular forces confined to molecular distances*. The disregard of this cooperation is, strictly speaking, only temporarily allowable, namely so long as the conditions are such that its influence is either nothing or may be regarded as vanishingly small. But just as little as an objection to the law of gravitation is derived from this fact, ought any objection to the fundamental law of electrical action to be derived from the physically inadmissible conditions to which, according to Helmholtz, this law leads, when it is considered that these inadmissible conditions are connected only with certain molecular distances.

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## XVI. *Notices respecting New Books.*

*Theory of Heat. By* J. Clerk Maxwell, *M.A., F.R.S., Professor of Experimental Physics in the University of Cambridge.* London: Longmans, Green, and Co. 1871. (Pp. 312.)

THE subject of this work is correctly indicated by its title; it is a treatise on the *Theory* of Heat; its writer's aim having been to state and enforce the general propositions that have been established regarding the nature and effects of heat, rather than to discuss the particular facts which are summed up in those propositions. It must not be supposed, however, that no notice is taken of the experiments which form the basis of our knowledge of Heat; on the contrary, they are described, where necessary, at sufficient length to bring out the principles involved in them; but they are described